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MATHÉMATIQUES

LE PROBLÈME INVERSE DU CALCUL DES VARIATIONS

Par I. M. RAPOPORT

(Présenté par S. N. Bernstein, de l'Académie, le 3. XII. 1937)

Dans ce travail nous examinons le problème suivant:
Nous avons un système d'équations différentielles:

$$f_i\left(x, y_j, \overset{j=n}{\underset{j=1}{y'_j, y''_j}}\right) = 0, \quad i = 1, 2, \ldots, n, \tag{1}$$

vérifiant les conditions aux limites:

$$y_j(a_1) = b_{1j}, y_j(a_2) = b_{2j}, \quad j = 1, 2, \ldots, n; \tag{2}$$

en considérant comme lignes admissibles la réunion de toutes les courbes $y_1 = y_1(x), \ldots, y_n = y_n(x)$ de la classe C_2, réunissant les points $A(a_1, b_{11}, \ldots, b_{1n})$ et $B(a_2, b_{21}, \ldots, b_{2n})$ dans l'espace à $n+1$ dimensions $x, y_1, \ldots, y_n$, et telles que les fonctions $y_1(x), \ldots, y_n(x)$ sont uniformes dans l'intervalle (a_1, a_2), nous devons trouver dans cette classe $I(y_1, \ldots, y_n)$ la fonction de ligne qui remplit la condition suivante:

si la ligne admissible $y_1 = \bar{y}_1(x), \ldots, y_n = \bar{y}_n(x)$ réalise l'extrêmum de la fonctionnelle I, le système des fonctions $\bar{y}_1(x), \ldots, \bar{y}_n(x)$ forme une solution des équations différentielles (1) avec les conditions aux limites (2).

En vertu d'un lemme connu du calcul des variations la fonctionnelle I, dont la variation est:

$$\delta I = \int_{a_1}^{a_2} \sum_{i=1}^{n} f_i\left(x, y_j, \overset{j=n}{\underset{j=1}{y'_j, y''_j}}\right) \delta y_i \, dx, \tag{3}$$

donnera une solution de la question posée.

Ainsi, le problème inverse du calcul des variations amène à un des problèmes essentiels de l'analyse fonctionnelle—à la construction de la fonctionnelle d'après sa variation. Nous citons les conditions nécessaires et suffisantes pour l'existence d'une fonctionnelle qui possède une variation donnée, et montrons une méthode simple et commode pour la construire. Si l'intégrale définie cherchée existe, la fonction sous le signe d'intégrale peut être trouvée au moyen d'une seule quadrature. Nous allons expliquer notre méthode.

Supposons qu'il existe une fonctionnelle dont la variation est définie par l'égalité (3). Prenons deux courbes arbitraires admissibles:

$y_1 = \eta_1(x), \ldots, y_n = \eta_n(x)$ et $y_1 = y_1(x), \ldots, y_n = y_n(x)$, et cherchons la différence $I(y_1, \ldots, y_n) - I(\eta_1, \ldots, \eta_n)$. A cet effet formons une réunion de $m+1$ courbes admissibles:

$$y_1 = y_{1k}(x), \ldots, y_n = y_{nk}(x), \quad k = 0, 1, \ldots, m, \tag{4}$$

où la première ligne est la ligne $y_1 = \eta_1(x), \ldots, y_n = \eta_n(x)$, la dernière ligne—la ligne $y_1 = y_1(x), \ldots, y_n = y_n(x)$ [c'est-à-dire $y_{j0}(x) \equiv \eta_j(x)$, $y_{jm} \equiv y_j(x)$. $j = 1, 2, \ldots, n$], et $m-1$ autres lignes sont arbitraires. Formons la somme des variations de la fonctionnelle I, correspondant aux passages successifs de la ligne $k-1$ à la ligne k de la réunion (4):

$$S_m = \sum_{k=1}^{m} \int_{a_1}^{a_2} \sum_{i=1}^{n} f_i\left(x, y_{jk}, \overset{j=n}{\underset{j=1}{y'_{jk}}}, y''_{jk}\right) \delta y_{ik}\, dx, \tag{5}$$

où $\delta y_{ik}(x) = y_{ik}(x) - y_{i,k-1}(x)$.

Faisons maintenant augmenter indéfiniment le nombre m, pour que chaque différence $\delta y_{ik}(x)$ $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m)$ tende uniformément vers zéro en même temps que sa dérivée, dans l'intervalle (a_1, a_2). Alors la suite

$$S_1, S_2, \ldots, S_m, \ldots \tag{6}$$

tendra vers la différence $I(y_1, \ldots, y_n) - I(\eta_1, \ldots, \eta_n)$, quelle que soit la loi d'après laquelle la réunion des courbes (4) correspond à chaque nombre m. En considérant la différence trouvée $I(y_1, \ldots y_n) - I(\eta_1, \ldots, \eta_n)$ comme fonction de la «limite supérieure» de la ligne $y_1 = y_1(x), \ldots, y_n = y_n(x)$, nous trouvons la solution du problème posé. La fonctionnelle cherchée est définie à une constante près, cette constante peut être trouvée en connaissant la valeur d'une fonctionnelle pour l'une quelconque des lignes admissibles.

En calculant la limite S de la suite (6) nous arrivons au théorème suivant, que nous allons démontrer synthétiquement.

Théorème. *Si le système des fonctions*:

$$f_i\left(x, y_j, \overset{j=n}{\underset{j=1}{y'_j}}, y''_j\right), \quad i = 1, 2, \ldots, n,$$

satisfait aux conditions:

$$\left.\begin{aligned} \frac{\partial f_i}{\partial y_j} - \frac{d}{dx}\frac{\partial f_i}{\partial y'_j} + \frac{d^2}{dx^2}\frac{\partial f_i}{\partial y''_j} &\equiv \frac{\partial f_j}{\partial y_i} \\ -\frac{\partial f_i}{\partial y'_j} + 2\frac{d}{dx}\frac{\partial f_i}{\partial y''_j} &\equiv \frac{\partial f_j}{\partial y'_i} \\ \frac{\partial f_i}{\partial y''_j} &\equiv \frac{\partial f_j}{\partial y''_i} \end{aligned}\right\}, \quad \begin{matrix} i = 1, 2, \ldots, n, \\ j = 1, 2, \ldots, n, \end{matrix} \tag{A}$$

qui sont des conditions nécessaires pour l'existence d'une fonction $L\left(x, y_j, \overset{j=n}{\underset{j=1}{y'_j}}\right)$, *satisfaisant aux identités*:

$$f_i \equiv \frac{\partial L}{\partial y_i} - \frac{d}{dx}\frac{\partial L}{\partial y'_i}, \quad i = 1, 2, \ldots, n, \tag{B}$$

alors l'intégrale multiple:

$$S = \int_{a_1}^{a_2} \int_{\tau_1}^{\tau_2} \sum_{i=1}^{n} f_i\left[x, z_j(x, t), \overset{j=n}{\underset{j=1}{z'_{jx}}}(x, t), z''_{jxx}(x, t)\right] z'_{it}(x, t)\, dt\, dx, \tag{C}$$

conserve une valeur constante pour chaque système des fonctions $z_j(x,t)$, *uniformes et continues, ainsi que leurs dérivées partielles* $z'_{jx}, z'_{jxx}, z'_{jt}, z'_{jxxt}$ *à l'intérieur du domaine de l'intégration et satisfaisant aux conditions aux limites*:

$$z_j(a_1,t) \equiv b_{1j}, z_j(a_2,t) \equiv b_{2j}, z_j(x,\tau_1) \equiv$$
$$\equiv \eta_j(x), z_j(x,\tau_2) \equiv y_j(x), j = 1,2,\ldots,n. \qquad (D)$$

La valeur de l'intégrale multiple S est déterminée par deux courbes: $y_1 = \eta_1(x), \ldots, y_n = \eta_n(x)$ et $y_1 = y_1(x), \ldots, y_n = y_n(x)$, réunissant les points $A(a_1, b_{11}, \ldots, b_{1n})$ et $B(a_2, b_{21}, \ldots, b_{2n})$. Prenons comme classe des lignes admissibles la réunion de toutes les courbes $y_1 = y_1(x), \ldots, y_n = y_n(x)$ de la classe C_2, réunissant les points A et B pour lesquelles les fonctions $y_1(x), \ldots, y_n(x)$ sont uniformes dans l'intervalle (a_1, a_2), et considérons l'intégrale S comme une fonction de ligne $y_1 = y_1(x), \ldots, y_n = y_n(x)$, où la ligne $y_1 = \eta_1(x), \ldots, y_n = \eta_n(x)$ est considérée comme fixe.

Alors, si la ligne admissible $y_1 = \bar{y}_1(x), \ldots, y_n = \bar{y}_n(x)$ réalise l'extrêmum de l'intégrale S, le système des fonctions $\bar{y}_1(x), \ldots, \bar{y}_n(x)$ forme une solution du système des équations différentielles:

$$f_i\left(x, y_j, \overset{j=n}{\underset{j=1}{y'_j}}, y''_j\right) = 0, \quad i = 1,2,\ldots,n, \qquad (E)$$

avec les conditions aux limites:

$$y_j(a_1) = b_{1j}, \quad y_j(a_2) = b_{2j}, \quad j = 1,2,\ldots,n.$$

Démonstration. Pour démontrer que les conditions (A) sont des conditions nécessaires pour l'existence de la fonction L, il suffit de trouver les valeurs des f_i des identités (B) et les substituer dans (A).

Pour démontrer l'affirmation suivante du théorème, considérons l'expression pour la variation δS.

Introduisons la notation:

C — est un contour rectiligne dans le plan x,t avec les sommets:

$$K(a_1,\tau_1), \quad L(a_2,\tau_1), \quad M(a_2,\tau_2) \quad \text{et} \quad N(a_1,\tau_2),$$

Q — est le domaine limité par le contour C.

Alors:

$$\delta S = \iint_Q \sum_{i=1}^{n}\sum_{j=1}^{n}\Big[\Big(f'_{iz_j} - \frac{\partial}{\partial x} f'_{iz'_{jx}} + \frac{\partial^2}{\partial x^2} f'_{iz''_{jxx}} - f'_{jz_i}\Big) z'_{it} +$$
$$+\Big(-f'_{iz'_{jx}} + 2\frac{\partial}{\partial x} f'_{iz''_{jxx}} - f'_{jz'_{ix}}\Big) z''_{ixt} + \Big(f'_{iz''_{jxx}} - f'_{jz''_{ixx}}\Big) z'''_{ixxt}\Big] \delta z_j \, dx\,dt +$$
$$+ \int_C \sum_{i=1}^{n}\sum_{j=1}^{n} f'_{iz'_{jx}} z'_{it} \delta z_j \, dt + \int_C \sum_{i=1}^{n}\sum_{j=1}^{n} f'_{iz''_{jxx}} z'_{it} \delta z'_{jx} \, dt -$$
$$- \int_C \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x}\left(f'_{iz''_{jxx}} z'_{it}\right) \delta z_j \, dt - \int_C \sum_{i=1}^{n} f_i \delta z_i \, dx.$$

L'intégrale multiple dans la dernière égalité disparaît grâce à la condition (A), les trois premières intégrales curvilignes disparaissent parce que sur les côtés LM et NK du rectangle $KLMN$ les variations δz_j et les dérivées z'_{it} deviennent nulles en vertu des conditions (D). Ainsi

$$\delta S = -\int_C \sum_{i=1}^{n} f_i \delta z_i \, dx.$$

Si nous prenons deux lignes $y_1 = \eta_1(x), \ldots, y_n = \eta_n(x)$ et $y_1 = y_1(x), \ldots y_n = y_n(x)$ fixes, les variations δz_i seront aussi nulles sur les côtés KL et MN du rectangle $KLMN$.

Dans ce cas, la variation de l'intégrale multiple S sera nulle pour chaque système des fonctions $z_j(x, t)$ qui satisfont aux conditions du théorème, et, par conséquent, la valeur de l'intégrale multiple S ne dépendra réellement pas du choix de ce système des fonctions.

Considérons maintenant l'intégrale multiple S comme fonction de la ligne $y_1 = y_1(x), \ldots y_n = y_n(x)$, en supposant la ligne $y_1 = \eta_1(x), \ldots, y_n = \eta_n(x)$ fixe, comme il a été fait précédemment. Alors:

$$\delta S = -\int_M^N \sum_{i=1}^n f_i \, \delta z_i \, dx = -\int_{a_1}^{a_2} \sum_{i=1}^n f_i \left[x, z_j(x, \tau_2), z'_{jx}(x, \tau_2), \overset{j=n}{\underset{j=1}{}}, z''_{jxx}(x, \tau_2)\right] \times$$

$$\times \, \delta z_i(x, \tau_2) \, dx = \int_{a_1}^{a_2} \sum_{i=1}^n f_i \left(x, y_j, \overset{j=n}{\underset{j=1}{y'_j, y''_j}}\right) \delta y_i \, dx.$$

Si le système des fonctions $\overline{y}_1(x), \ldots, \overline{y}_n(x)$ réalise l'extrêmum de l'intégrale S, chaque fonction $f_i\left(x, y_j, \overset{j=n}{\underset{j=1}{y'_j, y''_j}}\right)$ est identiquement nulle pour ce système, donc il forme une solution du système des équations différentielles (E).

Le théorème est complètement démontré.

Pour calculer l'intégrale multiple S il est commode de poser:

$$\tau_1 = 0, \quad \tau_2 = 1, \quad z_j(x, t) = \eta_j(x) + t\,[y_j(x) - \eta_j(x)].$$

Nous arrivons à la formule:

$$S = \int_{a_1}^{a_2} \int_0^1 \sum_{i=1}^n f_i \left[x, \eta_j + t\,(y_j - \eta_j), \eta'_j + t \overset{j=n}{\underset{j=1}{(y'_j - \eta'_j)}}, \eta''_j + \right.$$

$$\left. + \, t\,(y''_j - \eta''_j)\right](y_i - \eta_i)\, dt\, dx. \qquad (7)$$

L'intégrale multiple S exprime la différence $I(y_1, \ldots, y_n) - I(\eta_1, \ldots, \eta_n)$ entre les valeurs de la fonctionnelle cherchée $I(y_1, \ldots, y_n)$ sur la courbe $y_1 = y_1(x), \ldots, y_n = y_n(x)$ et sur la courbe $y_1 = \eta_1(x), \ldots, y_n = \eta_n(x)$. En effectuant en (7) l'intégration par rapport à t, nous obtiendrons le résultat sous la forme:

$$I(y_1, \ldots y_n) - I(\eta_1, \ldots, \eta_n) = \int_{a_1}^{a_2} L^*\left(x, y_j, \overset{j=n}{\underset{j=1}{y'_j, y''_j}}\right) dx \Bigg|_{y_j = \eta_j(x)}^{y_j = y_j(x)}.$$

L'intégrale définie $\int_{a_1}^{a_2} L^*\left(x, y_j, \overset{j=n}{\underset{j=1}{y'_j, y''_j}}\right) dx$ est la solution particulière du problème posé, pour laquelle les équations d'Euler sont identiques aux équations différentielles données, les conditions (A) sont les conditions nécessaires pour l'existence d'une telle solution particulière. L'ordre de la fonction sous le signe d'intégrale dans la solution trouvée peut être diminué au moyen d'intégration par parties. Si, par exemple, nous avons l'équation:

$$y'' + k^2 y = 0, \quad y(a_1) = b_1, \quad y(a_2) = b_2.$$

on a:

$$I(y) - I(\eta) = \int_{a_1}^{a_2} \int_0^1 \{\eta'' + t(y'' - \eta'') + k^2[\eta + t(y - \eta)]\}(y - \eta)\, dt\, dx =$$

$$= \frac{1}{2} \int_{a_1}^{a_2} [y'' + \eta'' + k^2(y + \eta)](y - \eta)\, dx =$$

$$= \frac{1}{2} \int_{a_1}^{a_2} [-y'^2 + \eta'^2 + k^2(y^2 - \eta^2)]\, dx +$$

$$+ \frac{1}{2}(y' + \eta')(y - \eta)\Big|_{a_1}^{a_2} = \int_{a_1}^{a_2} \left(-\frac{y'^2}{2} + \frac{k^2 y^2}{2}\right) dx \Big|_{y=\eta(x)}^{y=y(x)},$$

parce que

$$\eta(a_1) = y(a_1), \quad \eta(a_2) = y(a_2).$$

Nous avons généralisé les résultats exposés dans cet article pour les équations aux dérivées partielles et pour les équations intégrales. D'ailleurs, nous avons examiné en détail la question de la construction de la solution la plus générale du problème inverse du calcul des variations. En considérant les équations de la dynamique des systèmes matériels, nous avons obtenu une nouvelle construction des principes de variations dans la mécanique.

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MATHÉMATIQUES

SUR QUELQUES PROPRIÉTÉS DES FONCTIONS RÉGULIÈRES

Par A. BERMANT

(Présenté par I. M. Vinogradow, de l'Académie, le 3. XII. 1937)

1. En s'appuyant sur le principe de Grötzsch et ses conséquences, G. M. Golousin [1] a démontré les théorèmes suivants:

A. Si $w = f(z) = z + a_2 z^2 + \ldots$ est régulière dans $|z| < 1$, il existe n (n est un entier positif quelconque) segments rectilignes, sortant du point $w = f(0) = 0$, formant des angles égaux, entièrement contenus dans la représentation du cercle $|z| < 1$, réalisée par cette fonction, et tels que la somme de leurs longueurs est aussi proche de n que l'on veut.

B. Si $w = f(z) = z + a_2 z^2 + \ldots$ est régulière dans $|z| < 1$, l'aire de l'étoile de la représentation du cercle $|z| < 1$ par la fonction $w = f(z)$ est au moins égale à π.

Nous donnons ici le nom «d'étoile de la représentation» au plus grand domaine étoilé par rapport à $w = 0$ qui est contenu dans le domaine couvert par les valeurs de la fonction $w = f(z)$ et situé sur la feuille de la surface riemannienne de cette fonction qui contient le point $w = f(0) = 0$.

Dans la présente note je donne une méthode tout à fait différente et qui permet de préciser les théorèmes A et B.

2. Démontrons d'abord la proposition simple qui suit:

Lemme. *Si $w = f(z) = z + a_2 z^2 + \ldots$ est régulière et univalente dans un domaine fermé D contenant le point $z = 0$, on a:*

$$\int_\lambda \ln r \, d\theta = \int_l \ln \rho \, d\varphi, \tag{1}$$

où $z = r e^{i\varphi}$, $w = \rho e^{i\theta}$, l est la courbe qui forme la frontière de D, λ — la courbe décrite par $w = f(z)$ quand z décrit la courbe l. D'ailleurs, si $\rho = \rho(\theta)$ est l'équation de λ on suppose que la fonction $\rho(\theta)$ est une fonction univalente à dérivée finie.

Démonstration. On a le long de λ: $dw = w\left[i + \frac{\rho'(\theta)}{\rho(\theta)}\right] d\theta$, donc

$$\int_\lambda \ln \frac{w}{z} \left(i + \frac{\rho'}{\rho}\right) d\theta = 0.$$

Il en résulte:

$$\int_\lambda \ln \frac{\rho}{r} \, d\theta = \int_\lambda \frac{\rho'}{\rho} (\varphi - 0) \, d\theta,$$

et en intégrant par parties la seconde intégrale on obtient l'égalité cherchée.

Remarquons maintenant que l'aire du domaine qu'on obtient en représentant le domaine borné par λ au moyen de la fonction $\ln\frac{w}{z}$ est non négative:

$$\int_{\lambda} \ln\frac{\rho}{r}\, d(\theta - \varphi) \geqslant 0;$$

l'intégrale étant prise dans la direction positive.

En tenant compte de (1) on obtient

$$\int_{\lambda} \ln\rho\, d\theta \geqslant 2\int_{\lambda} \ln r\, d\theta - \int_{l} \ln r\, d\varphi. \qquad (2)$$

3. Théorème 1. *Si* $w = f(z) = z + a_2 z^2 + \ldots$ *est régulière dans* $|z|<1$, *la somme de l'aire de l'étoile de la représentation du cercle* $|z|<1$ *par la fonction* $w = f(z)$ *et du domaine contenu dans* $|z|<1$ *qui correspond à l'étoile est au moins égale à* 2π.

Démonstration. Supposons que $w = f(z)$ est univalente dans $|z|<1$. Soit λ_{r_0} la courbe qui donne la représentation de la circonférence $|z| = r_0 < 1$ au moyen de la fonction $w = f(z)$. Il est évident que le contour $\lambda^*_{r_0}$ de l'étoile de la représentation du cercle $|z| \leqslant r_0$ est formé d'arcs de la courbe λ_{r_0} et de segments des tangentes à cette courbe, menées par le point $w = 0$. Dans le plan des z il correspond à la courbe $\lambda^*_{r_0}$ une courbe $l^*_{r_0}$ formée d'arcs de la circonférence $|z| = r_0$ et d'un nombre fini de courbes $c_1, c_2, \ldots, c_k$ entièrement contenues dans le cercle $|z| \leqslant r_0$; les extrémités de chaque courbe c_i sont situées sur la circonférence $|z| = r_0$.

En prenant un contour assez proche à $\lambda^*_{r_0}$, en remplaçant les segments des tangentes à λ_{r_0} par des arcs des courbes de manière à vérifier les conditions du lemme, on voit que l'inégalité (2) subsiste pour $\lambda^*_{r_0}$.

D'ailleurs on a:

$$\int_{\lambda^*_{r_0}} \ln\rho\, d\theta \geqslant 2\cdot 2\pi \ln r_0 - \int_{l^*_{r_0}} \ln r\, d\varphi.$$

En ce qui concerne l'intégrale du second membre de l'inégalité, il est facile de voir que l'on a:

$$\int_{l^*_{r_0}} \ln r\, d\varphi = 2\pi \ln r_0 - \int_{\gamma} \ln r\, d\varphi,$$

la dernière intégrale étant prise, dans la direction positive, suivant le contour γ du domaine δ qu'on obtient en enlevant du cercle $|z| \leqslant r_0$ le domaine borné par la courbe $l^*_{r_0}$. Mais on a:

$$\int_{\gamma} \ln r\, d\varphi = \iint_{\delta} \left|\frac{1}{z}\right|^2 dx\, dy \geqslant \frac{1}{r_0^2}(\pi r_0^2 - s^*_{r_0}); \quad z = x + iy,$$

où $s^*_{r_0}$ est l'aire du domaine borné par la courbe $l^*_{r_0}$.

Ainsi

$$\int_{\lambda^*_{r_0}} \ln\rho\, d\theta \geqslant 2\pi \ln r_0 + \frac{1}{r_0^2}(\pi r_0^2 - s^*_{r_0}),$$

ou bien

$$\frac{1}{2\pi}\int_{\lambda^*_{r_0}} \ln\rho\, d\theta \geqslant \ln r_0 + \frac{1}{2\pi r_0^2}(\pi r_0^2 - s^*_{r_0}). \qquad (3)$$

Il en résulte

$$e^{\frac{1}{2\pi}\int\limits_{\lambda^*_{r_0}} \ln \rho^2 d\theta} \geqslant r_0^2 e^{\left(1-\frac{s^*_{r_0}}{\pi r_0^2}\right)} \geqslant r_0^2\left(2-\frac{s^*_{r_0}}{\pi r_0^2}\right).$$

En vertu d'une relation connue entre la moyenne arithmétique et la moyenne géométrique, nous trouvons

$$\frac{1}{2}\int\limits_{\lambda^*_{r_0}} \rho^2 d\theta \geqslant \pi r_0^2\left(2-\frac{s^*_{r_0}}{\pi r_0^2}\right) = 2\pi r_0^2 = s^*_{r_0}$$

c'est-à-dire,

$$S^*_{r_0} + s^*_{r_0} \geqslant 2\pi r_0^2,$$

où $S^*_{r_0}$ est l'aire de l'étoile de la représentation du cercle $|z| \leqslant r_0$ par la fonction $w = f(z)$. En faisant tendre r_0 vers l'unité, on obtient l'inégalité cherchée.

Si la fonction n'est pas univalente, la démonstration reste la même; il faut seulement faire passer des coupures radiales par les points de ramification de la surface riemannienne de la fonction.

En cherchant une borne inférieure pour l'intégrale

$$\int\limits_{\lambda^*_{r_0}} \ln \frac{\rho}{r}\, d(\theta - \varphi)$$

on peut encore préciser l'évaluation de l'aire de l'étoile de la représentation. Par exemple, l'inégalité suivante est vraie

$$S_1^* + s_1^* \geqslant 2\pi + |a_2|^2 S_1,$$

où S_1 est l'aire du cercle univalent ayant pour centre le point $w = f(0) = 0$ et couvert par les valeurs de la fonction.

4. Théorème II. *Si* $w = f(z) = z + a_2 z^2 + \ldots$ *est régulière dans* $|z| < 1$, *il existe n* (*n étant un entier positif quelconque*) *segments rectilignes sortant du point* $w = f(0) = 0$, *formant des angles donnés, situés entièrement dans la représentation du cercle* $|z| < 1$ *et tels que la somme de leurs longueurs est aussi proche de* σ *que l'on veut, où* $\sigma \geqslant n$.

Démonstration. Il suit de l'inégalité (3) que l'on a:

$$\frac{1}{2\pi}\int\limits_{\lambda^*_{r_0}} \rho d\theta \geqslant r_0 \frac{1}{2}\left(3-\frac{s^*_{r_0}}{\pi r_0^2}\right).$$

Il en résulte que pour un m assez grand

$$\sum_1^m \rho(\theta_i) \geqslant m \cdot \frac{1}{2}\left(3-\frac{s_1^*}{\pi}\right) \tag{4}$$

où les nombres θ_i $(i = \overline{1, m})$ forment sur l'intervalle $(0,2\pi)$ un système de nombre également distribué (gleichverteilt) et $\rho(\theta_i)$ est la longueur du segment issu du point $w = f(0) = 0$ mené jusqu'au contour de l'étoile λ_1^* et formant un angle égal à θ_i avec la direction positive de l'axe réel.

Menons par le point $w = f(0) = 0$ n segments formant des angles donnés (ces angles peuvent être supposés incommensurables deux à deux), et faisons tourner ce faisceau de segments autour de son sommet en conservant la somme des longueurs des segments égale à L; supposons que, dans chaque position de ce faisceau, l'un au moins des segments qui le forment couvre des points situés hors de l'étoile de la représentation;

on suppose d'ailleurs que tous les segments sont prolongés jusqu'au contour λ_1^*.

En faisant tourner le faisceau d'un angle égal à $\frac{2\pi}{p}, 2\cdot\frac{2\pi}{p}, 3\cdot\frac{2\pi}{p}, \ldots,$ $(p-1)\frac{2\pi}{p}$, nous nous trouverons, pour p assez grand, dans les conditions de l'inégalité (4) et d'ailleurs $m = pn$. Donc

$$Lp > \sum_1^{pn} \rho(\theta_i) \geqslant pn \cdot \frac{1}{2}\left(3 - \frac{s_1^*}{\pi}\right)$$

d'où il suit

$$L \quad n \cdot \frac{1}{2}\left(3 - \frac{s_1^*}{\pi}\right).$$

Ainsi

$$\sigma = n \cdot \frac{1}{2}\left(3 - \frac{s_1^*}{\pi}\right).$$

En faisant une évaluation plus précise on peut, par exemple, obtenir

$$\sigma = n \cdot \frac{1}{2}\left(3 - \frac{s_1^*}{\pi}\right)\left(1 + \frac{|a_2|^2 S_1}{2\pi}\right).$$

Remarque. La méthode que nous venons d'exposer peut être étendue sans difficultés au cas d'un domaine multiplement connexe.

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MATHÉMATIQUES

SUR LE PROBLÈME D'OSCILLATION POUR L'ÉQUATION DIFFÉRENTIELLE LINÉAIRE DU DEUXIÈME ORDRE

Par M. IELCHIN

(Présenté par S. N. Bernstein, de l'Académie, le 11. XII. 1937)

1. La solution générale de l'équation différentielle:

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0, \tag{1}$$

peut être exprimée dans la forme suivante:

$$x = \frac{Re^{-\frac{1}{2}\int_T^t p(\xi)\,d\xi}}{\sqrt{\omega}} \cos\left\{\int_T^t \omega(\xi)\,d\xi + \gamma\right\}, \tag{2}$$

R et γ étant des constantes; $\omega(t)$ sera nommée «fréquence variable de l'oscillation» d'une solution particulière de l'équation canonique correspondant à (1):

$$\ddot{z} = I(t)z. \tag{3}$$

La fréquence $\omega(t)$ satisfait à l'équation différentielle:

$$-\frac{1}{2}\frac{\ddot{\omega}}{\omega} + \frac{3}{4}\left(\frac{\dot{\omega}}{\omega}\right)^2 - \omega^2 = I(t), \tag{4}$$

où

$$I(t) = \frac{\dot{p}}{2} + \frac{p^2}{4} - q \tag{5}$$

est l'invariant de l'équation (1).

2. Si

$$u(t) = \frac{1}{\sqrt{\omega}} \cos\int_T^t \omega\,d\xi; \quad v(t) = \frac{1}{\sqrt{\omega}} \sin\int_T^t \omega\,d\xi \tag{6}$$

désignent deux solutions particulières de l'équation canonique (3), l'expression:

$$\omega(t) = \frac{1}{u^2 + v^2} \tag{7}$$

est la solution générale de l'équation (4).

Les constantes d'intégration: ω_0 et $\dot{\omega}_0$ sont déterminées par les conditions initiales:

$$u|_{t=T} = u_0 \neq 0; \quad \dot{u}|_{t=T} = \dot{u}_0. \tag{8}$$

Les conditions initiales pour $\varphi(t)$ sont:

$$\varphi|_{t=T} = 0; \quad \dot{\varphi}|_{t=T} = \frac{1}{u_0}. \tag{9}$$

Comme

$$u\dot{\varphi} - \varphi\dot{u} = 1 \tag{10}$$

pour toute valeur de t, il résulte de (7) que sur chaque segment ne contenant aucun point singulier de l'invariant, toutes les fréquences sont positives et ne peuvent être égales ni à 0, ni à ∞.

3. Bohl [1] a obtenu le premier la solution de l'équation (1) sous la forme (2) en étudiant les équations à coéfficients quasi-périodiques. Plus tard plusieurs auteurs ont appliqué la même formule à quelques autres équations de la forme (1).

Nous allons appliquer cette forme de la solution au problème d'oscillation.

Supposons que l'intervalle $T \leqslant t < \infty$ ne contient aucun point singulier de l'invariant $I(t)$. D'après le dernier résultat du n° 2, les valeurs de t correspondant aux zéros des solutions, sont déterminées par la formule:

$$\int_T^t \omega(\xi)\, d\xi + \gamma = \frac{\pi}{2} + n\pi, \tag{11}$$

où $\omega(t)$ est une fréquence quelconque.

Deux cas peuvent se présenter par rapport à $\omega(t)$:

1) Pour toutes les valeurs de t il existe une constante A telle que

$$\int_T^t \omega(\xi)\, d\xi < A. \tag{12}$$

Toutes les solutions n'ont alors qu'un nombre fini d'oscillations.

2) Quelque grand que soit n, on peut indiquer une telle valeur T_n que pour $t > T_n$ on a:

$$\int_T^t \omega(\xi)\, d\xi > n\pi. \tag{13}$$

L'équation dans ce dernier cas est oscillatoire.

La solution classique du problème d'oscillations est basée sur le célèbre théorème de Sturm. La méthode développée dans le présent travail, est basée sur l'évaluation de la phase qui correspond à l'invariant donné $I(t)$; elle nous permet de résoudre la question pour tous les cas classiques et même pour quelques équations qui possèdent un invariant de signe variable et qui ne sont pas résolubles par les méthodes classiques.

4. Pour obtenir une relation entre la variation de la phase et celle de l'invariant, formons pour l'équation (4) l'équation aux variations:

$$\delta\ddot{\omega} - 3\frac{\dot{\omega}}{\omega}\delta\dot{\omega} + \left[-\frac{\ddot{\omega}}{\omega} + 3\left(\frac{\dot{\omega}}{\omega}\right)^2 + 4\omega^2\right]\delta\omega = -2\omega\dot{\delta I}. \tag{14}$$

En posant

$$\delta\omega = \omega^{\frac{3}{2}}\delta z \tag{15}$$

nous ramenons l'équation (14) à la forme canonique:

$$\ddot{\delta z} = \left[-\frac{1}{2}\frac{\ddot{\omega}}{\omega} + \frac{3}{4}\left(\frac{\dot{\omega}}{\omega}\right)^2 - 4\omega^2\right]\delta z - \frac{2\dot{\delta I}}{\sqrt{\omega}}. \tag{16}$$

Pour l'équation homogène correspondant à (16), l'équation des fréquences (4) devient:

$$-\frac{1}{2}\frac{\ddot{\lambda}}{\lambda} + \frac{3}{4}\left(\frac{\dot{\lambda}}{\lambda}\right)^2 - \lambda^2 = -\frac{1}{2}\frac{\ddot{\omega}}{\omega} + \frac{3}{4}\left(\frac{\dot{\omega}}{\omega}\right)^2 - 4\omega^2, \tag{17}$$

et l'une des fréquences est

$$\lambda = 2\omega. \tag{18}$$

Donc la solution de (14) aux conditions initiales:

$$\delta\omega\big|_{t=T} = 0; \quad \dot{\delta\omega}\big|_{t=T} = 0 \tag{19}$$

est de la forme

$$\delta\omega = -\omega(t)\int_T^t \frac{\delta I(s)}{\omega(s)}\sin 2\int_s^t \omega(\xi)\,d\xi\,ds. \tag{20}$$

Nous obtenons la formule (20) si nous résolvons l'équation (16) par la méthode de la variation des constantes, et si nous exprimons δz au moyen de $\delta\omega$ par la formule (15).

5. La formule (20) a lieu pour la famille des invariants:

$$\bar{I}(t, \alpha) = I(t) + \alpha[\bar{I}(t) - I(t)], \quad 0 \leqslant \alpha \leqslant 1. \tag{21}$$

En désignant par $\bar{\omega}(t, \alpha)$ l'une des fréquences correspondant à l'invariant $\bar{I}(t, \alpha)$, nous aurons:

$$\dot{\bar{\omega}}(t, \alpha) = -\bar{\omega}(t, \alpha)\int_T^t \frac{\delta\alpha\,[\bar{I}(s) - I(s)]}{\bar{\omega}(s, \alpha)}\sin 2\int_s^t \bar{\omega}(\xi, \alpha)\,d\xi\,ds. \tag{22}$$

En intégrant par rapport au paramètre α entre 0 et 1, nous aurons

$$\omega(t) - \bar{\omega}(t) = \int_T^t [\bar{I}(s) - I(s)]\int_0^1 \frac{\bar{\omega}(t, \alpha)}{\bar{\omega}(s, \alpha)}\sin 2\int_s^t \bar{\omega}(\xi, \alpha)\,d\xi\,d\alpha\,ds, \tag{23}$$

où $\bar{\omega}(t)$ désigne une fréquence correspondant à l'invariant $\bar{I}(t)$.

Intégrons l'expression (23) entre T et t:

$$\int_T^t \omega(y)\,dy - \int_T^t \bar{\omega}(y)\,dy =$$

$$= \int_T^t dy\int_T^y [\bar{I}(s) - I(s)]\int_0^1 \frac{\bar{\omega}(y, \alpha)}{\bar{\omega}(s, \alpha)}\sin 2\int_s^y \bar{\omega}(\xi, \alpha)\,d\xi\,d\alpha\,ds. \tag{24}$$

En permutant l'ordre des intégrations d'après la formule de Dirichlet, nous arrivons au théorème des phases:

$$\int_T^t \omega(y)\,dy - \int_T^t \bar{\omega}(y)\,dy = \int_0^1 d\alpha \int_T^t \frac{[\bar{I}(s) - I(s)]}{\bar{\omega}(s,\tau)} \sin^2 \int_s^t \bar{\omega}(\xi,\alpha)\,d\xi\,d\alpha. \quad (25)$$

Dans le cas particulier, où l'invariant $I(t)$ varie d'une manière monotone, la phase varie aussi d'une manière monotone, mais dans le sens inverse. Ce théorème permet d'élucider complètement la nature du théorème classique de Sturm.

6. En appliquant le théorème des phases au problème de l'oscillation pour l'équation (1), il est utile de considérer les invariants de comparaison:

$$\bar{I}(t) = \varphi(t) + \Big(\int \varphi dt\Big)^2 - k^2 e^{-4\iint \varphi dt^2}. \quad (26)$$

Les constantes de l'intégrale double sont fixées; on obtient l'intégrale simple en différentiant l'intégrale double.

L'équation des fréquences (4) correspondant à l'invariant (26) est

$$-\frac{1}{2}\frac{\ddot{\omega}}{\omega} + \frac{3}{4}\Big(\frac{\dot{\omega}}{\omega}\Big)^2 - \omega^2 = \varphi(t) + \Big(\int \varphi dt\Big)^2 - k^2 e^{-4\iint \varphi dt^2}. \quad (27)$$

Un calcul direct montre que

$$\bar{\omega}(t) = k e^{-2\iint \varphi dt^2} \quad (28)$$

est l'une des fréquences correspondant à l'invariant (26).

Puisque une solution particulière de l'équation

$$\ddot{y} = \Big[\varphi + \Big(\int \varphi dt\Big)^2\Big] y \quad (29)$$

est

$$y = e^{\iint \varphi dt^2}, \quad (30)$$

l'équation (29) ne peut être oscillatoire, quelle que soit la fonction $\frac{1}{\varphi(t)}$ intégrable dans tout segment $T \leqslant t \leqslant A < \infty$. Donc l'équation dont l'invariant est donné par la formule (26) ne peut être oscillatoire pour $k = 0$; il en est de même à plus forte raison, pour k imaginaire.

Ainsi les conditions pour que l'équation ayant l'invariant (26) soit oscillatoire, sont les suivantes:

$$1)\quad k \text{ est réel et } \neq 0,$$

$$2)\quad \int_T^\infty e^{-2\iint \varphi dt^2} dt = \infty. \quad (31)$$

En substituant dans la formule (26) au lieu de φ des fonctions élémentaires quelconques, nous aurons les invariants de comparaison avec des fréquences (28).

Posons, en particulier,

$$\varphi = -\frac{1}{2t^2}, \quad (32)$$

à l'aide du théorème des phases nous arrivons au critère connu de Kneser (2).

Soit

$$\varphi(t) \approx \sum_{i=1}^{\infty} (a_i \cos \lambda_i t + b_i \sin \lambda_i t) \tag{33}$$

une fonction presque-périodique sans terme constant dont les deux premières intégrales indéfinies sont presque-périodiques pour certaines constantes d'intégration. En déterminant k par la condition:

$$k^2 e^{-4 \int\int \varphi dt^2} < \lambda^2, \tag{34}$$

nous voyons que toutes les équations

$$\ddot{y} = [-\lambda^2 + \varphi(t)] y \tag{35}$$

sont oscillatoires, car d'après (26) et (34)

$$\overline{I}(t) > -\lambda^2 + \varphi(t) \tag{36}$$

et, en vertu du théorème des phases,

$$\int_T^{\infty} \omega(y)\, dy > k \int_T^{\infty} e^{-2 \int\int \varphi dt^2}\, dt = \infty. \tag{37}$$

Si pour toutes les valeurs de $t > T$ l'invariant d'une certaine équation ne surpasse pas un invariant quelconque de $-\lambda^2 + \varphi(t)$, cette équation est oscillatoire en vertu du même théorème des phases.

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ASTROPHYSICS

THE PROBLEM OF RADIATIVE EQUILIBRIUM OF STELLAR ATMOSPHERES FOR THE COEFFICIENT OF ABSORPTION DEPENDING ON THE FREQUENCY

By E. R. MUSTEL

(Communicated by V. G. Fessenkov, Member of the Academy, 28. XI. 1937)

Consider the model of a stellar atmosphere in which the radiation is supposed to consist of two streams parallel to the axis h; let I_ν be the intensity of the outward stream at the point h, where the temperature is T, I'_ν, the inward stream. Then, the equation of transfer may be reduced to the following form:

$$\frac{d\left\{\int_0^\infty I_\nu\, d\nu + \int_0^\infty I'_\nu\, d\nu\right\}}{d\xi} = \frac{3}{2}\int_0^\infty (I_\nu - I'_\nu)\, k_\nu\, d\nu \qquad (1)$$

where

$$d\xi = \rho dh \qquad (2)$$

and ρ is the density; k_ν, the coefficient of continuous absorption; dh, the element of the radius-vector from the given point of the atmosphere to the centre of the star. The condition for radiative equilibrium for the model under consideration is

$$2\int_0^\infty B_\nu\, k_\nu\, d\nu = \int_0^\infty (I_\nu + I'_\nu)\, k_\nu\, d\nu, \qquad (3)$$

B_ν the intensity of black-body radiation at the local temperature T.

We postulate that intensities I_ν and I'_ν may be represented as Planck's functions with unknown temperatures T_1 and T_2.

This supposition may be justified by the fact that the variation of the coefficient of absorption with frequency is much more important than the violation of Planck's shape of curves of I_ν and I'_ν due to this variation *.

* When $k_\nu = k = \text{const}$ the curves of I_ν and I'_ν are approximate copies of Planck's curves.

This supposition is the base of the present method. Thus:

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT_1}} - 1}; \tag{4a}$$

$$I'_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT_2}} - 1} \tag{4b}$$

where T_1 and T_2 are temperatures till unknown.

For a thin boundary layer the expression (4b) has to be altered as in this layer I'_ν is approximately proportional to $\tau_\nu = \int k_\nu d\xi$, being in its spectral composition nearly constant *.

The computation of T_2 made by the author for the model with $k_\nu = k = \text{const}$ according to the formula:

$$\frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT_2}} - 1} = e^{-t} \int_0^\infty B_\nu(t)\, e^t dt \tag{5}$$

where

$$T^4 = T_0^4 (1 + 2t) \tag{5a}$$

showed that the depth of this layer $\tau = \tau_0$ does not exceed 0.15.

This result is of a general character and will not be altered much by computations based on the coefficient of absorption depending on frequency.

Thus, up to $\tau = \tau_0$ we may, without making a noticeable error, put

$$I'_\nu = \zeta(T) \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}. \tag{6}$$

Taking account of (4a), (4b) and (6), the condition of the constancy of the net-flux $F = \sigma T_e^4$ will be

$$T_1^4 - \zeta(T) T^4 = T_e^4 \text{ from } \tau = 0 \text{ up to } \tau = \tau_0, \tag{7a}$$

$$T_1^4 - T_2^4 = T_e^4 \text{ from } \tau = \tau_0 \text{ up to } \tau = \infty. \tag{7b}$$

The coefficient of continuous absorption for normal stellar atmospheres may be represented as product of two functions:

$$k_\nu = \Phi(\xi, T)\, \Psi(\nu, T). \tag{8}$$

In (8) we took into consideration the fact that, according to Pannekoek [1], processes of absorption are determined by the local temperature T.

Owing to (8), the condition for radiative equilibrium (3) will be

$$[2 - \zeta(T)] F(T) = F(T_1) \text{ from } \tau = 0 \text{ up to } \tau = \tau_0, \tag{9a}$$

$$2F(T) = F(T_1) - F(T_2) \text{ from } \tau = \tau_0 \text{ up to } \tau = \infty \tag{9b}$$

where

$$F(T) = \frac{2h}{c^2} \int_0^\infty \Psi(\nu, T) \frac{\nu^3 d\nu}{e^{\frac{h\nu}{kT}} - 1}; \tag{10}$$

$$F(T_i) = \frac{2h}{c^2} \int_0^\infty \Psi(\nu, T) \frac{\nu^3 d\nu}{e^{\frac{h\nu}{kT_i}} - 1} \qquad (i = 1, 2). \tag{11}$$

* This fact has been kindly pointed out to the author by prof. Ambarzumian.

From two pairs of equations (7a), (9a) and (7b), (9b) we find $\zeta(T), T_1 T_2$ as functions of T. At $\tau = \tau_0$ there will be a little jump in T.

From the condition $\zeta(T_0) = 0$ we determine the boundary temperature T_0:

$$2F(T_0) = F(T_e). \qquad (12)$$

Now the equation of transfer (1) will be as follows:

$$\frac{dT}{d\xi} = \frac{3\pi}{16\sigma} \frac{F(T_1) - \zeta F(T)}{T_1^3} \frac{dT}{dT_1} \Phi(\xi, T) \text{ from } \tau = 0 \text{ up to } \tau = \tau_0, \qquad (13a)$$

$$\frac{dT}{d\xi} = \frac{3\pi}{16\sigma} \frac{F(T_1) - F(T_2)}{T_1^3} \frac{dT}{dT_1} \Phi(\xi, T) \text{ from } \tau = \tau_0 \text{ up to } \tau = \infty. \qquad (13b)$$

The right-hand sides of these differential equations are known functions of ξ and T.

The jump in $\frac{dT}{d\xi}$ at $\tau = \tau_0$ is quite unimportant.

The solution of equations (13a) and (13b) with the corresponding initial condition gives the temperature T as a function of ξ.

Further we find k_ν as a function of ξ, this gives $\tau_\nu = \int k_\nu d\xi$.

Having found τ_ν we can determine for each point of atmosphere the values of I_ν and I'_ν for different frequencies in the spectrum. If the variation of the latter values with frequency is very great (the limit of Balmer's series for A_0 stars) we may consider the obtained distribution of the temperature as the first approximation. Indeed, knowing I_ν and I'_ν as a function of ν and ξ and taking into consideration that an error in the distribution of temperature does not influence these values very much, we find $\int_0^\infty I_\nu d\nu$, $\int_0^\infty I'_\nu d\nu$, $\int_0^\infty I_\nu k_\nu d\nu$, $\int_0^\infty I'_\nu k_\nu d\nu$ and solve the following differential equation, which is the consequence of the equation (1). The right-hand side of this equation is a known function of ξ:

$$\frac{dT}{d\xi} = \frac{3}{2} \frac{\int_0^\infty (I_\nu - I'_\nu) k_\nu d\nu}{\frac{d}{dT}\left\{ \int_0^\infty I_\nu d\nu + \int_0^\infty I'_\nu d\nu \right\}}. \qquad (14$$

Having solved (14) we find a second approximation to the distribution of temperature.

A few remarks concerning the functions $\Phi(\xi, T)$ and $\Psi(\nu, T)$.

Their form depends, essentially, on the thermal conditions dominating in the atmosphere of a star under consideration. The electron pressure, entering into $\Phi(\xi, T)$ is by conditions of ionization related to the total gas pressure which, in turn, owing to the equation of hydrostatic equilibrium is proportional to ξ. The pressure of radiation is not taken into account. The degree of ionization x also will be a function of T. For hot stars, where the rate of ionization $x \approx 1$, we have thus

$$\Phi(\xi, T) \sim \frac{\xi}{T^{3/2}}. \qquad (15)$$

The equations (13a) and (13b) are in this case equations with separated variables.

For cool stars account has to be taken of the question concerning the abundance of different absorbing elements.

For hot stars the function $\Psi(\nu, T)$ is characterized by a series of maxima which are situated in places corresponding to the limits of hydrogen series. For cool stars may be used the expression, which is found by the smoothing of the combination of individual bands. To the difference in the form of function $\Psi(\nu, T)$ corresponds the difference in the form of equations (9a) and (9b).

Note that the present method indirectly gives the theory of photosphere, so far as the temperature T is obtained as a function of ξ, and not the optical depth τ.

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MECHANIK

ZUR DEFINITION DER STABILITÄTSWAHRSCHEINLICHKEIT

Von V. STEPANOFF

(Vorgelegt von S. N. Bernstein, Mitglied der Akademie, am 11. XII. 1937)

In zwei Artikeln ([1], [2]) hat N. D. Moisseiev den Begriff der Stabilitätswahrscheinlichkeit eingeführt. Diese sinngemässe Erweiterung des Begriffes der Stabilität im Liapounoffschen Sinne, hat ihren Wert, wie der Verfasser selbst in ([1]) bemerkt, in der Tatsache, dass aus dem Vorhandensein von nur einer Bewegung in der Umgebung des Gleichgewichtspunktes, die für $0 \leqslant t < \infty$ nicht auf eine beliebig kleine Umgebung dieses Punktes beschränkt bleibt, folgt, dass derselbe im Liapounoffschen Sinne labil ist; indessen ist es höchst unwahrscheinlich, dass bei zufälligen Störungen der Anfangsbedingungen die gestörte Bewegung mit dieser labilen zusammenfällt; praktisch muss also in diesem Falle die Gleichgewichtslage als stabil betrachtet werden.

Die etwas umständliche, vom Verfasser gegebene Definition der Grösse, welche die Stabilitätswahrscheinlichkeit charakterisiert, veranlasst mich, eine beinahe äquivalente und meiner Ansicht nach viel einfachere Definition vorzuschlagen.

1. Wir betrachten das System

$$\frac{dx_s}{dt} = X_s(x_1, x_2, \ldots, x_n), \quad s = 1, 2, \ldots, n,$$

wo $X_s(0, 0, \ldots, 0) = 0$ ist und die Funktionen X_s in einer Umgebung U des Koordinatenursprungs O stetig und so beschaffen sind, dass in U jeder Anfangspunkt $P(x_1^0, x_2^0, \ldots, x_n^0)$ eine einzige Lösung $x_1(t), x_2(t), \ldots, x_n(t)$, oder kürzer $f(P, t)$ bestimmt. Eine solche Lösung $f(P, t)$ nennen wir eine Bewegung und betrachten dieselbe für $t \geqslant 0$. Wir führen die Bezeichnung ein:

$$\sup_{0 \leqslant t < \infty} \rho(f(P, t), O) = \overline{R}(P).$$

Die Stabilitätswahrscheinlichkeit der trivialen Lösung

$$x_1 = x_2 = \ldots = x_n = 0$$

wird von uns folgendermassen definiert:

Wir beschreiben um O zwei Kugeln mit den Halbmessern r und R. $S(r)$ bzw. $S(R)$; $r < R$, $S(R) \subset U$. Mit $E(R, r)$ wird die Menge derjenigen Punkte $P \in S(r)$ bezeichnet, für welche $\overline{R}(P) < R$ ausfällt. (Die Bewegungen, für welche $\overline{R}(P) < R$ ist, nennen wir R-stabil.)

Die mittlere Dichte von $E(R, r)$ in $S(r)$ ist

$$\frac{\mathfrak{M}\, E(R, r)}{\mathfrak{M}\, S(r)}, \tag{1}$$

wobei $\mathfrak{M}$ das Lebesguesche n-dimensionale Mass bezeichnet.

Führen wir die Grösse

$$\lim_{r \to 0} \frac{\mathfrak{M}\, E(R, r)}{\mathfrak{M}\, S(r)} = PS(R) \tag{2}$$

ein, vorausgesetzt, dass der Limes für alle genügend kleine R existiert. Die Grösse (2) charakterisiert die R-Stabilität im Punkte O. Da der Zähler in (1) eine nicht abnehmende Funktion von R ist, so folgt aus der Existenz von (2) immer die Existenz von

$$\lim_{R \to 0} PS(R) = PS. \tag{3}$$

Die Zahl PS nennen wir die Stabilitätswahrscheinlichkeit der ungestörten (trivialen) *Lösung.*

Anmerkung. Anstatt der Kugel $S(r)$ und der Menge $E(R, r)$ können wir derer Durchschnitt mit einer messbaren Menge M nehmen. Dann betrachten wir anstatt der Ausdrücke (2), (3) die folgenden:

$$\lim_{r \to 0} \frac{\mathfrak{M}(E(R, r) \cdot M)}{\mathfrak{M}(S(r) \cdot M)} = PS_M(R), \tag{2'}$$

$$\lim_{R \to 0} PS_M(R) = PS_M \tag{3'}$$

(vorausgesetzt, dass der Nenner in (2') nicht verschwindet). Die Grösse PS_M nennen wir *relative Stabilitätswahrscheinlichkeit in bezug auf die Menge M*. Diese Bemerkung wird uns im folgenden von Nutzen sein.

2. Wir wollen unsere Definition mit der Moisseievschen vergleichen. Er betrachtet nämlich die Mengen M, die O als einen Häufungspunkt besitzen, und führt die Grösse ein:

$$\sup_{P \in M \cdot S(r)} \overline{R}(P) = \overline{R}(r, M).$$

Falls

$$\lim_{r \to 0} \overline{R}(r, M) = 0, \tag{S}$$

dann gehört die Menge M zur S-Klasse (stabil) und wird mit MS bezeichnet.

Wir wollen zeigen, dass, wenn $\mathfrak{M}(MS \cdot S(r)) > 0$ für jedes $r > 0$, die relative Wahrscheinlichkeit $PS_{MS} = 1$ ist.

Aus der Relation (S) folgt, dass es sich für ein gegebenes $R > 0$ eine Zahl $\delta(R)$ so angeben lässt, dass für $r < \delta(R)$, $\overline{R}(r, MS) < R$ ist, das heisst, die Mengen $E(R, r) \cdot MS$ und $S(r) \cdot MS$ sind identisch. Hieraus folgt wegen (2') und (3'):

$$PS_{MS} = 1.$$

Falls die Menge MS im Punkte O eine bestimmte Dichte $d \geqslant 0$ besitzt, das heisst

$$\lim_{r \to 0} \frac{\mathfrak{M}(MS \cdot S(r))}{\mathfrak{M}\, S(r)} = d,$$

so erhalten wir aus (2) und (3):

$$\lim_{R\to 0}\lim_{r\to 0}\frac{\mathfrak{M}(E(R,r)\cdot MS)}{\mathfrak{M}S(r)}=d,$$

also bringt MS in die Grösse PS den Betrag d.

Ferner gehört nach Moisseiev die Menge M zur I-Klasse (labil), falls für dieselbe die Limesgleichung (S) nicht zutrifft; unter den Mengen dieser Klasse sind von Bedeutung die irreduziblen MI, das heisst solche, die keine Untermenge der S-Klasse enthalten.

Wir wollen die Struktur einer irreduziblen MI-Menge untersuchen. Wir nehmen eine Zahlenfolge

$$R_1>R_2>\ldots>R_k>\ldots,\quad \lim_{k\to\infty}R_k=0.$$

Bezeichnen wir mit M_k die Menge der Punkte $P\in MI$, für welche $\overline{R}(P)>R_k$ ausfällt. Es ist

$$M_1\subset M_2\subset\ldots\subset M_k\subset\ldots\subset MI.$$

Ich behaupte, dass man ein N und ein $r_0>0$ finden kann derart, dass die Mengen $M_N\cdot S(r_0)$ und $MI\cdot S(r_0)$ zusammenfallen. Sonst könnte man einen Punkt P_1 finden derart, dass $\rho(O,P_1)<r_1=\frac{R_1}{2}$ und $\overline{R}(P_1)<R_1$ ist. Im allgemeinen sei $R_{n_k}<r_{k-1}$, wir setzen $r_k=\frac{1}{2}R_{n_k}$; da die Menge $S(r_k)\cdot(MI-M_{n_k})$ nach Voraussetzung nicht leer ist, könnte man einen Punkt $P_k\in S(r_k)$ finden, so dass $\overline{R}(P_k)<R_{n_k}$ ist.

Die Menge $\Sigma=\{P_k\}$ hat O zum Limespunkt und hat die Eigenschaft, dass $\lim_{r\to 0}\overline{R}(r,\Sigma)=0$, also ist $\Sigma\subset MI$ eine Menge der Klasse S, was der Irreduzibilität der MI widerspricht.

Daraus folgt: für jede irreduzible Menge MI kann man ein $r_0>0$ und ein $R_0>0$ finden, so dass $\overline{R}(P)>R_0$ für $P\in S(r_0)\cdot MI$ ist. Also ist $E(R_0,r_0)\cdot MI=0$, und falls $\mathfrak{M}(S(r)\cdot MI)>0$ für $r>0$, finden wir aus (2′) und (3′):

$$PS_{MI}=0.$$

Die Menge MI trägt also nichts zu PS bei.

Nach diesen Bemerkungen kann man leicht zeigen, dass in allen von N. D. Moisseiev betrachteten Fällen [(²) S. 515—519] die von ihm definierte Stabilitätswahrscheinlichkeit denselben Wert hat, der durch unsere Formeln (2) und (3) gegeben wird. In der Tat handelt es sich in allen jenen Fällen um die Bestimmung der Dichte im Punkte O einer Menge MS (oder MS^*, siehe unten). Und es ist: entweder

$$S(r)=MS+MI, \tag{A}$$

oder

$$S(r)=MS^*+MI^*, \tag{B}$$

wobei die Mengen MS^* und MI^* sich durch die MS- und MI-Mengen approximieren lassen, und zwar gleichmässig in bezug auf die mittlere Dichte; endlich gibt es Fälle, wo die Zerlegungen (A) und (B) der Kugel $S(r)$ nur bis auf eine Menge der Dichte Null im Punkte O möglich

sind. *In allen diesen Fällen ist der Betrag der Stabilitätswahrscheinlichkeit durch unsere Formeln* (2) *und* (3) *gegeben*, und die in Frage stehenden Limites existieren.

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MÉCANIQUE

SUR LES SYSTÈMES DYNAMIQUES STABLES AU SENS DE LIAPOUNOFF

Par M. BEBOUTOFF

(Présenté par S. N. Bernstein, de l'Académie, le 11. XII. 1937)

Nous donnons le nom de système dynamique M à un espace topologique où l'on a défini un groupe monoparamétrique $f(p, t)$ $(p \in M, -\infty < t < +\infty)$ de transformations de M en lui-même qui satisfait aux conditions suivantes:

1) $f(p, 0) = p$,

2) $f[f(p, t_1), t_2] = f(p, t_1 + t_2)$,

3) si $p_n \to p$ et $t_n \to t$, on a $f(p_n, t_n) \to f(p, t)$.

L'ensemble des points $f(p, t), p$ étant fixe, $-\infty < t < +\infty$, est nommé trajectoire, nous la désignerons par $f(p, t)$, le point p est son point initial.

L'ensemble des points $f(p, t), p$ fixe, $t \geqslant 0$ $(t \leqslant 0)$, est nommé sémi-trajectoire positive (négative) avec le point initial p; nous les désignerons par $f^+(p, t)[f^-(p, t)]$. Une trajectoire $f(p, t)$ est positivement (négativement) *stable au sens de Lagrange* s'il existe un ensemble compact F tel que $F \supset f^+(p, t)[F \supset f^-(p, t)]$.

Dans le cas contraire la trajectoire $f(p, t)$ est positivement (négativement) instable au sens de Lagrange. Une trajectoire stable (instable) au sens de Lagrange tant positivement que négativement est nommée *trajectoire stable* (instable)*.

Un *système dynamique* M est nommé *stable* (instable) si toutes les trajectoires de M sont stables (instables).

Nous dirons que le point p est *stable au sens de Liapounoff*, si pour tout $\varepsilon > 0$ il existe un $\delta(p, \varepsilon) > 0$ tel que, si

$$\rho(p, q) < \delta,$$

on a pour $-\infty < t < +\infty$

$$\rho[f(p, t), f(q, t)] < \varepsilon.$$

Il est évident que tout point d'une trajectoire $f(p, t)$ est stable au sens de Liapounoff, si le point p est stable.

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* Il est évident que la propriété de stabilité est indépendente du choix du point initial: c'est une propriété de la trajectoire entière.

Si tous les points de M sont stables au sens de Liapounoff, nous dirons que le *système M est stable au sens de Liapounoff.*

Nous appellerons points limites dynamiques de $f(p,t)$ les points limites alpha et oméga au sens de Birkhoff [1].

Nous supposerons partout dans la suite que M est un *sous-espace fermé et connexe de E^n*, et que le système dynamique *M est stable au sens de Liapounoff.*

Théorème 1. *Si le système M contient une trajectoire stable positivement ou négativement au sens de Lagrange, alors le système M est stable.*

Supposons pour préciser que $f(p,t)$ est positivement stable au sens de Lagrange.

Il est connu [1] que dans ce cas l'ensemble des points limites oméga de $f(p,t)$ contient une trajectoire $f(p_0,t)$ qui est stable.

Il est aisé d'établir que (en vertu de la stabilité au sens de Liapounoff) l'ensemble M_1, qui est composé de tous les points des trajectoires stables, et l'ensemble M_2, qui est composé de tous les points des trajectoires instables positivement ou négativement, sont des ensembles ouverts, et que l'on a

$$M = M_1 + M_2, \quad M_1 \cdot M_2 = 0, \quad M_1 \supset f(p_0,t).$$

Alors, en vertu de la connexité de M, on a $M_2 = 0$, et le théorème est démontré.

Corollaire. *Si le système M contient une trajectoire instable positivement ou négativement au sens de Lagrange, le système est instable.*

Théorème 2. *Si $q \in \overline{f(p,t)}$, on a $\overline{f(q,t)} = \overline{f(p,t)}$.* Cela résulte immédiatement du fait que si $f(p,t_n)$ tend vers q, alors $f(q,-t_n)$ tend vers p.

Corollaire. *Si le système M est stable, chaque trajectoire de M appartient à un certain ensemble minimal. Par conséquent elle est presque périodique, périodique ou représente un point de repos* [2], *et M est une réunion d'ensembles minimaux.*

Considérons maintenant le cas où le système M est instable.

Théorème 3. *Si M est un système dynamique instable, aucune trajectoire n'a de points limites dynamiques.* Admettons le contraire. Supposons qu'il existe une trajectoire $f(p_0,t)$ telle que $f(p_0,t_n) \to q_0$ pour $t_n \to \infty$.

Soit $\rho(p_0,q_0) = R$, nous pouvons supposer $R > 0$.

En vertu de l'instabilité de $f(p_0,t)$, il existe une suite de points $\{f(p_0,\tau_n)\}$, telle que $\tau_n \to \infty$ et que

$$\rho[p_0, f(p_0,\tau_n)] > 4nR.$$

Définissons maintenant ϑ_n tel que $\rho[p_0, f(p_0,\vartheta_n)] = 4R$ et que, pour $\vartheta_n < t \leqslant \tau_n$:

$$\rho[p_0, f(p_0,t)] > 4R.$$

Comme $\{f(p_0,\vartheta_n)\} \subset \overline{S(p_0,4R)}$, la suite $\{f(p_0,\vartheta_n)\}$ contient une suite convergente.

Supposons pour ne pas compliquer la notation que

$$f(p_0,\vartheta_n) \to q_1.$$

Il est évident que l'ensemble des nombres $\{(\tau_n - \vartheta_n)\}$ est non borné, car dans le cas contraire on pourrait supposer que $(\tau_n - \vartheta_n) \to T$ et alors

$$f(p_0,\tau_n) = f[f(p_0,\vartheta_n), \tau_n - \vartheta_n] \to f(q_1,t),$$

ce qui n'est pas possible en vertu du choix de $\{f(p_0, \tau_n)\}$. En vertu de la stabilité de q_1 au sens de Liapounoff on peut choisir un nombre $\delta(q_1, R)$ et un n_0 tel que pour $n \geqslant n_0$:

$$\rho[q_1, f(p_0, \vartheta_n)] < \delta.$$

Soit maintenant $t > \vartheta_{n_0}$, prenons $n_1 > n_0$ tel que $0 < t - \vartheta_{n_0} < \tau_{n_1} - \vartheta_{n_1}$.

Nous avons:

$$\begin{aligned}\rho[f(p_0, t), q_0] \geqslant \rho[p_0, f(p_0, \vartheta_{n_1} + t - \vartheta_{n_0})] - \rho[p_0, q_0] - \\ - \rho[f(p_0, \vartheta_{n_1} + t - \vartheta_{n_0}), f(q_1, t - \vartheta_{n_0})] - \\ - \rho[f(q_1, t - \vartheta_{n_0}), f(p_0, \vartheta_{n_0} + t - \vartheta_{n_0})] > \\ > 4R - R - R - R = R.\end{aligned}$$

Donc pour tout $t > \vartheta_{n_0}$ on a $\rho[f(p_0, t), q_0] > R$, ce qui est impossible parce que $\rho[f(p_0, t_n), q_0] \to 0$ pour $t_n \to \infty$. Cette contradiction démontre notre théorème.

Suivant V. Niemytzki (3) nous dirons qu'un système dynamique M a «un col à l'infini», s'il existe un nombre K tel que pour tout $N > K$ on peut indiquer un point p et deux nombres τ et $t\,(0 < \tau < t)$ tels que

$$\rho[p_0, p] < K, \quad \rho[p_0, f(p, \tau)] > N, \quad \rho[p_0, f(p, t)] < K.$$

Théorème 4. *Si le système M est instable, il n'a pas de col à l'infini.*

Supposons le contraire, soit M un système instable ayant un col à l'infini.

Soit p_n, τ_n et $t_n\,(0 < \tau_n < t_n)$ tels que $\rho[p_0, p_n] < K$, $\rho[p_0, f(p_n, \tau_0)] > n$, $\rho[p_0, f(p_0, t_n)] < K$. Supposons pour ne pas compliquer notre notation que $p_n \to p^1$, $f(p_n, t_n) \to q^1$.

Deux cas sont possibles:

a) $|t_n| < L$; dans ce cas $|\tau_n| < L$, et il est possible de supposer que $\tau_n \to \tau$, donc on a $f(p_n, \tau_n) \to f(p_1, \tau)$, ce qui contredit le choix de $\{f(p_n, \tau_n)\}$.

b) $\{t_n\}$ est non borné; pour ne pas compliquer la notation supposons que $t_n \to \infty$.

Montrons que dans ce cas le point q^1 est un point limite dynamique de $f(p^1, t)$.

Soit $\varepsilon > 0$ un nombre donné. En vertu de la stabilité de p^1 au sens de Liapounoff on peut choisir un $\delta\left(p^1, \frac{\varepsilon}{2}\right)$ et un n_0 tel que pour tout $n \geqslant n_0$ les inégalités suivantes auront lieu:

$$\rho[p^1, p_n] < \delta \quad \rho[f(p_n, t_n), q^1] < \frac{\varepsilon}{2}.$$

Dans ce cas pour $n \geqslant n_0$ on a

$$\rho[f(p^1, t_n), q^1] \leqslant \rho[f(p^1, t_n), f(p_n, t_n)] + \rho[f(p_n, t_n), q^1] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

c'est-à-dire, on a $\lim\limits_{n\to\infty} f(p^1, t_n) = q^1$ et $\lim\limits_{n\to\infty} t_n = \infty$.

D'après le théorème 3 ce cas est aussi impossible. Le théorème est démontré.

Corollaire. *Un système dynamique M instable au sens de Lagrange est homéomorphe à un système de droites parallèles, car dans ce cas il est instable et n'a pas de col à l'infini* (3).

Des théorèmes 1 et 4 il suit le

Théorème fondamental. *Si le système dynamique M est un ensemble fermé et connexe de E^n et s'il est stable au sens de Liapounoff, on a les deux possibilités suivantes; le système M est une réunion d'ensembles minimaux, et toutes les trajectoires de M sont presque périodiques* (en particulier M peut contenir des trajectoires périodiques et des points de repos), *ou bien M est homéomorphe à un système de droites parallèles.*

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MECHANIK

ÜBER DIE BEWEGUNGSSTABILITÄT NACH DER ERSTEN NÄHERUNG

Von I. MALKIN

(Vorgelegt von S. N. Bernstein, Mitglied der Akademie, d. 11. XII. 1937)

I

Es seien

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + L_s(t, x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, n) \qquad (1)$$

die Differentialgleichungen einer gestörten Bewegung, die im Bereich

$$t \geqslant 0, \quad |x_s| \leqslant H \qquad (s = 1, 2, \ldots, n) \qquad (2)$$

ein einziges Cauchysches Integral besitzen. Die Funktionen L_s sollen im Bereich (2) den Ungleichungen

$$|L_s(t, x_1, \ldots, x_n)| < A(x_1^2 + \ldots + x_n^2)^m \qquad (a)$$

genügen, wobei A und m zwei positive Konstanten bedeuten, von denen die zweite grösser als $\frac{1}{2}$ ist. Die Grössen p_{sk} seien Funktionen von t, die für alle $t \geqslant 0$ eindeutig und stetig sind. Die Funktionen p_{sk} können sowohl beschränkt als auch unbeschränkt sein. Wir bezeichnen ferner mit $x_{sk}(t, t_1)$ $(s, k = 1, 2, \ldots, n)$ das Fundamentalsystem von Lösungen des Differentialgleichungssystems der ersten Näherung

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n,$$

welches durch die Anfangsbedingungen

$$x_{sk}(t_1, t_1) = 1, \qquad \text{für } s = k,$$
$$x_{sk}(t_1, t_1) = 0, \qquad \text{für } s \neq k,$$

bestimmt wird, wobei t_1 der Anfangsmoment bedeutet. Wir nehmen an, dass diese Lösungen lauter positive charakteristische Zahlen besitzen, d. h. dass es eine positive Zahl α derart gibt, dass für jedes $t \geqslant t_1$ die Ungleichungen

$$|x_{sk}(t, t_1)| < B(t_1)\, e^{-\alpha(t-t_1)} \qquad (b)$$

gelten, wobei die Grösse $B(t_1)$ im allgemeinen eine unbeschränkte Funktion von t_1 ist.

Es ist bekannt, dass die Bedingung, dass die charakteristischen Zahlen des Differentialgleichungssystems der ersten Näherung positiv sind, nicht hinreichend ist dafür, dass die ungestörte Bewegung stabil sei, wie auch die Zusatzfunktionen L_s gewählt sein mögen. Wir nehmen zusätzlich an, dass die Grösse $B(t_1)$ der Ungleichung:

$$B(t_1)\, e^{-\beta t_1} < M = \text{const}, \qquad \text{(c)}$$

genügt, wobei

$$0 \leqslant \beta < (2m-1)\,\alpha$$

ist, und wollen folgenden Satz beweisen:

Satz. *Genügen die Lösungen des Differentialgleichungssystems der ersten Näherung den Bedingungen* (b) *und* (c), *so ist die ungestörte Bewegung bei jeder Wahl der Zusatzfunktionen* L_s, *die den Bedingungen* (a) *genügen stabil, und dabei gilt*

$$\lim_{t\to\infty} x_s e^{\mu t} = 0,$$

falls

$$\mu < \frac{\beta}{2m-1}$$

ist, und die Anfangsstörungen genügend klein sind.

Beweis. Setzen wir

$$\lambda = \frac{\beta}{2m-1}, \quad y_s = x_s e^{\lambda t}, \qquad (3)$$

so nimmt das Differentialgleichungssystem (1) die Form:

$$\frac{dy_s}{dt} = p_{s1}y_1 + \dots + (p_{ss}+\lambda)\,y_s + \dots + p_{sn}y_n + e^{\lambda t}L_s(t, e^{-\lambda t}y_1, \dots, e^{-\lambda t}y_n) \qquad (4)$$

an. Bezeichnen wir nun mit $y_{sk}(t, t_1)$ die Lösungen des entsprechenden Linearsystems, die durch die Anfangsbedingungen:

$$y_{sk}(t_1, t_1) = 1, \quad \text{für } s = k,$$
$$y_{sk}(t_1, t_1) = 0, \quad \text{für } s \neq k,$$

bestimmt sind, so genügen die Funktionen y_{sk} den Ungleichungen:

$$|y_{sk}(t, t_1)| < B(t_1)\, e^{-(\alpha-\lambda)(t-t_1)}. \qquad (5)$$

Aus (4) folgt:

$$y_s(t) = \sum_{i=1}^{n} c_i y_{si} + \sum_{i=1}^{n} \int_{t_1}^{t} y_{si}(t, z)\, e^{\lambda z} L_i(z, e^{-\lambda z}y_1, \dots, e^{-\lambda z}y_n)\, dz, \qquad (6)$$

wobei die c_i willkürliche Anfangswerte (bei $t = t_1$) von y_s bedeuten. Es sei t ein beliebiger Zeitpunkt und

$$\left.\begin{aligned} \varepsilon &= \max\{|y_1(t)|, \dots, |y_n(t)|\}, \\ \eta &= \max\{|c_1|, \dots, |c_n|\} \end{aligned}\right\} \qquad (7)$$

Dann gelten für diesen Zeitpunkt wegen (5) und (a) die Ungleichungen:

$$|y_s| < n\eta B(t_1)\, e^{-(\alpha-\lambda)(t-t_1)} + nA\varepsilon^{2m} \int_{t_1}^{t} |B(z)|\, e^{-(\alpha-\lambda)(t-z)-(2m-1)\lambda z}\, dz$$

und also wegen (c):

$$|y_s| < n\eta B(t_1)\, e^{-(\alpha-\lambda)(t-t_1)} + nAM\varepsilon^{2m} \int_{t_1}^{t} e^{-(\alpha-\lambda)(t-z)}\, dz =$$

$$= n\eta B(t_1)\, e^{-(\alpha-\lambda)(t-t_1)} + \frac{nAM\varepsilon^{2m}}{\alpha-\lambda}\left(1 - e^{-(\alpha-\lambda)(t-t_1)}\right).$$

Da aber

$$t > t_1, \quad \alpha - \lambda = \alpha - \frac{\beta}{2m-1} > 0$$

ist, so folgt

$$|y_s(t)| < n\eta B(t_1) + \frac{nAM\varepsilon^{2m}}{\alpha-\lambda}. \tag{8}$$

Daraus ist leicht zu erkennen, dass die ungestörte Bewegung hinsichtlich der Grössen $y_1, \ldots, y_n$ stabil ist, d. h. dass es für jede beliebig kleine positive Zahl ε eine andere positive Zahl η gibt, derart, dass, falls die Anfangswerte von y_s (d. h. c_s) den Ungleichungen

$$|y_s(t_1)| \leqslant \eta$$

genügen, so für jedes $t \geqslant t_1$ die Ungleichungen:

$$|y_s(t)| < \varepsilon, \tag{9}$$

erfüllt sind. In der Tat—wir setzen

$$\eta < \varepsilon, \quad \eta < \frac{\varepsilon}{2nB(t_1)}. \tag{10}$$

und wählen die Grösse ε so klein, dass

$$\frac{nAM\varepsilon^{2m-1}}{\alpha-\lambda} < \frac{1}{2} \tag{11}$$

(dies ist immer möglich, da $2m-1>0$). Da $\eta < \varepsilon$, so genügen die y_s den Bedingungen (9) falls $t - t_1$ genügend klein ist. Da aber die y_s sich stetig mit t ändern, so können sie erst dann aufhören, den Bedingungen (9) zu genügen, wenn es einen Zeitpunkt t gibt, im welchen die Bedingung (7) befriedigt wird. Dann müssen aber auch die Bedingungen (8) erfüllt sein, die wegen (10) und (11) ergeben:

$$|y_s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

was mit (7) unverträglich ist. Wir müssen also schliessen, dass die Ungleichungen (9) für jedes $t > t_1$ gelten, d. h. dass die ungestörte Bewegung hinsichtlich der Grössen $y_1, \ldots, y_n$ stabil ist. Da aber die x_s mit den y_s durch (3) verbunden sind, so folgt ohne weiteres, dass die ungestörte Bewegung hinsichtlich x_s stabil ist und dass die Gleichungen:

$$\lim_{t\to\infty} x_s e^{\mu t} = 0$$

gelten, falls

$$\mu < \frac{\beta}{2m-1},$$

w. z. b. w.

Ist $\beta = 0$ und ist die Funktion $p_n + \ldots + p_{nn}$ beschränkt, so erhält man den Satz von K. Persidsky, den er in der Arbeit: «Über Stabilität einer Bewegung nach der ersten Näherung» (1) gewonnen hat.

Bemerkung. Man kann in den Bedingungen (a), welchen die Zusatzfunktionen L_s genügen, die Grösse A nicht als konstant, sondern als eine unbeschränkte Funktion von t annehmen, die der Bedingung

$$Ae^{-\gamma t} < K = \text{const}$$

genügt, wobei

$$\gamma < (2m-1)\alpha - \beta.$$

Die Bedingungen für die Stabilität in der ersten Näherung mit unbeschränkten Zusatzfunktionen L_s wurden zuerst von K. Persidsky [2] in der Arbeit: «Über Stabilität der Integrale von Differentialgleichungssystemen» behandelt. Das Kriterium, das er erhalten hat, ist von dem unserigen verschieden und kann auch dann erfüllt sein, wenn unsere Bedingungen nicht erfüllt sind und umgekehrt.

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ZITIERTE LITERATUR

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II]

VERALLGEMEINERUNG DES FUNDAMENTALSATZES VON LIAPUNOFF ÜBER DIE STABILITÄT DER BEWEGUNGEN

(Vorgelegt von S. N. Bernstein, Mitglied der Akademie, d. 11. XII. 1937)

Es seien

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) \quad (s = 1, 2, \ldots, n) \tag{1}$$

die Differentialgleichungen einer gestörten Bewegung, wobei die X_s im Bereich

$$t \geqslant 0, \quad |x_s| \leqslant H \quad (s = 1, 2, \ldots, n) \tag{2}$$

eindeutige, stetige und bei $x_1 = \ldots = x_n = 0$ verschwindende Funktionen von $t, x_1, \ldots, x_n$ sind. Wir wollen ausserdem voraussetzen, dass die Gleichungen (1) im Bereich (2) ein einziges Cauchysches Integral besitzen.

Wir betrachten im folgenden einige Funktionen von $t, x_1, \ldots, x_n$, die im Bereich (2) eindeutig und differenzierbar sind. Genügt eine solche Funktion $V(t, x_1, \ldots, x_n)$ der Ungleichung:

$$V \geqslant W(x_1, \ldots, x_n), \tag{3}$$

wobei $W(x_1, \ldots, x_n)$ eine von t unabhängige, im Bereich (2) positive und nur bei $x_1 = \ldots = x_n = 0$ verschwindende Funktion von $x_1, \ldots, x_n$ ist, so heisst die Funktion V definit-positiv. Gilt dagegen

$$V \leqslant -W(x_1, \ldots, x_n), \tag{4}$$

so heisst die Funktion V definit-negativ.

Wir sagen auch, dass eine Funktion $V(t, x_1, \ldots, x_n)$ eine unendlich kleine obere Grenze zulässt, falls für jede beliebig kleine positive Zahl l eine andere positive Zahl $h(l)$ angegeben werden kann derart, dass die Ungleichung:

$$|V(t, x_1, \ldots, x_n)| \geqslant l,$$

die Ungleichungen

$$\sum_{s=1}^{n} |x_s| \geqslant h$$

mit sich bringt.

Der Fundamentalsatz von Liapounoff über die Stabilität der Bewegungen, welcher der sogenannten zweiten Methode zugrunde liegt, kann bekanntlich folgendermassen formuliert werden:

Gibt es eine positiv-definite Funktion $V(t, x_1, \ldots, x_n)$, die eine unendlich kleine obere Grenze zulässt und deren totale Ableitung nach t infolge der Differentialgleichungen der gestörten Bewegung, d. h. der Ausdruck

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x} X_s , \qquad (5)$$

definit-negativ ist, so ist die ungestörte Bewegung asymptotisch stabil.

Wir wollen diesen Satz verallgemeinern und folgenden Satz beweisen:

Satz. *Gibt es für die Differentialgleichungen der gestörten Bewegung eine positiv-definite Funktion* $V(t, x_1, \ldots, x_n)$, *die eine unendlich kleine obere Grenze zulässt, und die die Eigenschaft besitzt, dass die Differenz U der totalen Ableitung* (5) *und einer negativ-definiten Funktion* V_1 *in jedem fixierten Bereich*

$$0 < \lambda \leqslant \sum_{s=1}^{n} |x_s| \leqslant H$$

gleichmässig gegen Null strebt, falls $t \to \infty$, *so ist die ungestörte Bewegung stabil.*

Beweis. Es sei t_0 der Anfangsmoment, und es seien x_s^0 die Anfangswerte von x_s. Es muss bewiesen werden, dass es für jede beliebig kleine positive Zahl ε eine andere positive Zahl η gibt, derart, dass falls die x_s^0 den Ungleichungen:

$$|x_s^0| \leqslant \eta \quad (s = 1, 2, \ldots, n),$$

genügen, für jedes $t \geqslant t_0$ die Ungleichungen

$$|x_s| \leqslant \varepsilon \quad (s = 1, 2, \ldots, n)$$

gelten müssen.

Wir haben den Bedingungen des Satzes zufolge:

$$\lim_{t \to \infty} U = \lim_{t \to \infty} \left[\frac{dV}{dt} - V_1 \right] = 0$$

$$\left[\text{gleichmässig im Bereich } \lambda \leqslant \sum_{s=1}^{n} |x_s| \leqslant (u) \right] \qquad (6)$$

$$V(t, x_1, \ldots, x_n) \geqslant W(x_1, \ldots, x_n), \qquad (7)$$

$$V_1(t, x_1, \ldots, x_n) \leqslant -W_1(x_1, \ldots, x_n), \qquad (8)$$

wobei W_1 und W_2 zwei von t unabhängige positiv-definite Funktionen von $x_1, \ldots, x_n$ sind.

Es sei ε eine beliebig kleine positive Grösse. Wir bezeichnen mit l die genaue untere Grenze von V, falls die Bedingung:

$$\max \{ |x_1|, \ldots, |x_n| \} = \varepsilon$$

erfüllt ist. Da (7) gilt, so ist die Grösse l von Null verschieden und positiv. Betrachten wir die Gesamtheit aller Werte von x_s, die der Bedingung

$$V = l \qquad (9)$$

genügen, so gilt offenbar:

$$|x_s| \leqslant \varepsilon, \quad \text{falls } V = l, \text{ und } |x_s| < \varepsilon, \quad \text{falls } V < l. \qquad (10)$$

Da aber V eine unendlich kleine obere Grenze zulässt, so gilt ausserdem

$$\sum_{s=1}^{n} |x_s| \geqslant \lambda, \quad \text{falls } V = l, \qquad (11)$$

wobei λ eine genügend kleine positive Grösse bezeichnet.

Wir betrachten jetzt die Grösse $\frac{dV_1}{dt}$ für alle Werte von t, x_s, die der Gleichung (9) genügen. Wegen (6), (8), (10) und (11), muss eine genügend grosse positive Zahl $T(\lambda, \varepsilon)$ derart existieren, dass für alle $t \geqslant T$ die Grösse $\frac{dV}{dt}$ negativ ist. Wir haben also:

$$\left[\frac{dV}{dt}\right]_{V=l} < 0, \quad \text{falls } t \geqslant T. \tag{12}$$

Wir betrachten jetzt die Grössen x_s als gewisse Funktionen von t, die den Differentialgleichungen (1) genügen. Da die rechten Seiten dieser Gleichungen bei $x_1 = \ldots = x_n = 0$ verschwinden, so kann eine positive Zahl η so klein gewählt werden, dass für alle Werte von t, die der Bedingung

$$t_0 \leqslant t \leqslant T$$

genügen, die Bedingung

$$V < l$$

ebenfalls erfüllt ist, wenn nur

$$|x_s^0| \leqslant \eta. \tag{13}$$

Wir nehmen an, dass die Grösse η wirklich den angedeuteten Bedingungen genügt. Dann ist aber leicht zu erkennen, dass für alle $t \geqslant t_0$ die Ungleichung

$$V \leqslant l \tag{14}$$

erfüllt sein muss. In der Tat wird die Ungleichung (14) für einen Zeitpunkt nicht erfüllt, so muss ein Zeitpunkt T_1 derart existieren, dass für $t = T_1$ $V = l$ und für $t = T_1 + \delta$ $V > l$ wird, wie klein auch die positive Grösse δ sein mag. Dann muss aber für $T = T_1$ die Ungleichung

$$\left[\frac{dV}{dt}\right]_{V=l} > 0$$

gelten, was mit (12) unverträglich ist, da offenbar $T_1 > T$. Wir müssen also schliessen, dass für alle $t \geqslant t_0$ die Ungleichung (14) gilt. Wegen (10) muss aber auch

$$|x_s| \leqslant \varepsilon$$

sein, w. z. b. w.

Beispiel. Wir betrachten die Gleichungen

$$\frac{dx_s}{dt} = \frac{1}{1+t}(p_{s1}x_1 + \ldots + p_{sn}x_n) - x_s^3 \quad (s = 1, 2, \ldots, n),$$

wobei die p_{sk} beliebige beschränkte Funktionen von t sind, und setzen

$$2V = x_1^2 + \ldots + x_n^2.$$

Dann erhalten wir

$$\frac{dV}{dt} = \frac{1}{1+t}\sum_{s=1}^{n}(p_{s1}x_1 + \ldots + p_{sn}x_n)x_s - \sum_{s=1}^{n} x_s^4,$$

woraus sich ergibt, dass V allen Bedingungen des bewiesenen Satzes genügt. Die ungestörte Bewegung ist also stabil.

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PHYSIQUE MATHÉMATIQUE

SUR LE PROBLÈME INVERSE DU POTENTIEL

Par P. NOVIKOFF

(Présenté par I. M. Vinogradow, de l'Académie, le 25. XI. 1937)

Soit $W(x, y, z)$ le potentiel extérieur d'une masse inconnue; que peut on dire sur la distribution de cette masse? Il est facile de voir que si l'on pose la question dans cette forme générale, on a: quel que soit le domaine G contenant tous les points singuliers du prolongement harmonique de la fonction W, il existe une distribution de densité $\mu(x, y, z)$ de la forme $\mu(x, y, z) = u(x, y, z) + h(x, y, z)$ et telle que $W = \iiint_G \frac{\mu}{r} d\tau$, $u(x, y, z)$ étant une fonction harmonique, bien définie, que l'on peut déterminer à partir du potentiel W et h — une fonction arbitraire orthogonale à toutes les fonctions harmoniques dans le domaine G. Donc, dans ce cas il n'existe pas de solution unique du problème posé. Cependant si l'on admet que le domaine G et la densité μ satisfassent à certaines conditions supplémentaires, la solution du problème posé devient unique. Par exemple, il est facile de démontrer, que si G est un domaine simplement connexe, dont la surface possède en chaque point un plan tangent, si $\mu(x, y, z) = 1$, et le potentiel extérieur $W = \frac{1}{r}$, le domaine G coïncide nécessairement avec une sphère déterminée.

Dans cette note nous donnons quelques conditions suffisantes pour que le problème posé ait une solution unique. Ces conditions sont contenues dans l'énoncé du théorème suivant.

Théorème. *Soit* $\mu(x, y, z)$ *une fonction définie dans tous les points de l'espace xyz, telle que* $\min \mu > 0$ *et que la variation totale de* μ *le long d'une droite arbitraire ne dépasse pas* $\min \mu$. *Soient* G_1 *et* G_2 *deux domaines étoilés par rapport à un point intérieur commun et remplis de masses* M_1 *et* M_2 *de densité* $\mu(x, y, z)$. *Si les masses* M_1 *et* M_2 *ont un même potentiel extérieur, les domaines* G_1 *et* G_2 *coïncident.*

Un théorème analogue a lieu pour le problème plan dans le cas du potentiel logarithmique. La démonstration devient particulièrement simple si l'on suppose que μ est une constante. Nous donnons ici la démonstration de ce cas. Soient G_1 et G_2 deux domaines plans étoilés par rapport à l'origine, et soient $r_1 = r_1(\varphi)$ et $r_2 = r_2(\varphi)$ les équations des contours de G_1 et G_2, $r_1(\varphi)$ et $r_2(\varphi)$ étant deux fonctions positives, continues, arbitraires. Les domaines G_1 et G_2 sont remplis de masses de densité $\mu = 1$ et les potentiels de ces deux masses coïncident dans le domaine extérieur.

Il faut démontrer que $r_1(\varphi) \equiv r_2(\varphi)$. Remarquons d'abord que si une masse quelconque possède un potentiel extérieur identiquement nul la densité de cette masse est une fonction $\nu(x, y)$ orthogonale à chaque fonction harmonique dans le domaine rempli de cette masse.

En effet, soit G le domaine rempli de la masse donnée et Γ une circonférence qui contient tout ce domaine. Appliquons la formule de Green à deux fonctions U et W, U étant une fonction harmonique, continue possédant des dérivées partielles continues aux points de Γ, et $W = \iint \nu \ln r \, d\sigma$:

$$\iint_G (U\Delta W - W\Delta U)\, d\sigma = \int_\Gamma \left(U \frac{\partial W}{\partial n} - W \frac{\partial U}{\partial n} \right) ds.$$

Mais W et $\frac{\partial W}{\partial n}$ sont nuls aux points de la circonférence Γ, et par suite:

$$\iint_G U\Delta W \, d\sigma = \iint_G U\nu \, d\sigma = 0.$$

Quelle que soit la fonction harmonique V bornée sur G, elle peut être approximée à l'intérieur de G au moyen d'une fonction harmonique U, ayant les dérivées partielles continues sur le contour de G, de telle façon que l'on ait $\iint_G |V - U| \, d\sigma < \varepsilon$ quel que soit $\varepsilon > 0$ et par suite

$$\iint_G \nu V \, d\sigma = 0.$$

Revenons maintenant à nos domaines G_1 et G_2. Il suit de notre remarque que pour démontrer la coïncidence de ces deux domaines il suffit de construire une fonction harmonique U bornée sur le domaine $G_1 + G_2$ et telle que $\iint_{G_1} U \, d\sigma - \iint_{G_2} U \, d\sigma$ diffère de zéro. Soient $R_1 = R_1(\varphi)$ et $R_2 = R_2(\varphi)$ les équations des contours des domaines $G_1 \cdot G_2$ et $G_1 + G_2$. On a:

$$\iint_{G_1} U \, d\sigma - \iint_{G_2} U \, d\sigma = \int_0^{2\pi} \eta(\varphi) N(\varphi) \, d\varphi$$

si l'on pose

$$N(\varphi) = \int_{R_1(\varphi)}^{R_2(\varphi)} U(\rho, \varphi) \, \rho \, d\rho \qquad (*)$$

et

$$\eta(\varphi) = 1 \quad \text{pour} \quad r_2(\varphi) < r_1(\varphi),$$
$$\eta(\varphi) = -1 \quad \text{pour} \quad r_2(\varphi) > r_1(\varphi).$$

Par suite notre proposition sera démontrée si nous construisons une fonction $N(\varphi)$ de la forme (*), telle que

$$\int_{\varphi_1}^{\varphi_2} N(\varphi) d\varphi > 0$$

et

$$\int_0^{\varphi_1} N(\varphi) \, d\varphi + \int_{\varphi_2}^{2\pi} N(\varphi) \, d\varphi = 0, \qquad (1)$$

l'intervalle (φ_1, φ_2), étant contenu dans l'ensemble des valeurs de la variable φ pour lesquelles on a $r_2(\varphi) > r_1(\varphi)$.

Considérons l'expression

$$\int_0^R U(\rho, \varphi)\, \rho\, d\rho,$$

U étant une fonction harmonique sur $G_1 + G_2$. Il est facile de démontrer que cette expression peut être mise sous la forme $R^2 V(R, \varphi)$, V étant une fonction harmonique sur $G_1 + G_2$. Donc nous pouvons écrire:

$$N(\varphi) = R_2^2 V(R_2, \varphi) - R_1^2 V(R_1, \varphi).$$

Si nous construisons une fonction V harmonique et bornée dans $G_1 + G_2$, telle que pour cette fonction $N(\varphi)$ satisfasse à la condition (1), notre théorème sera démontré, car dans ce cas $N(\varphi)$ peut être approximée au moyen de l'expression $\int_{R_1}^{R_2} U \rho d\rho$, U étant une fonction harmonique et bornée dans $G_1 + G_2$, et l'on aura:

$$\int_0^{2\pi} [\eta(\varphi) \int_{R_1}^{R_2} U \rho\, d\rho]\, d\varphi \neq 0.$$

Démontrons maintenant l'existence de la fonction V qui satisfait aux conditions posées. Désignons par Δ l'intervalle (φ_1, φ_2) et par I le complémentaire de Δ par rapport à l'intervalle $(0, 2\pi)$.

Construisons une suite de fonctions harmoniques

$$V_0(R, \varphi),\ V_1(R, \varphi), \ldots, V_n(R, \varphi) \ldots$$

satisfaisant aux relations suivantes

$$\left.\begin{aligned}
&V_0(R_2, \varphi) = 0, && \text{si } \varphi \subset I;\\
&V_0(R_2, \varphi) = \frac{q}{(\varphi_2 - \varphi_1) R_2^2}, && \text{si } \varphi \subset \Delta,\ q > 0;\\
&\vdots\\
&V_n(R_2, \varphi) = \frac{\int_I R_1^2 V_{n-1}(R_1, \varphi)\, d\varphi}{\int_I R_2^2\, d\varphi}, && \text{si } \varphi \subset I;\\
&V_n(R_2, \varphi) = \frac{\int_\Delta R_1^2 V_{n-1}(R_1, \varphi)\, d\varphi}{\int_\Delta R_2^2 d\varphi}, && \text{si } \varphi \subset \Delta.
\end{aligned}\right\} \qquad (2)$$

Les quantités

$$\frac{\int_\Delta R_1^2 d\varphi}{\int_\Delta R_2^2 d\varphi}$$

et

$$\frac{\int_I R_1^2 d\varphi}{\int_I R_2^2 d\varphi}$$

sont inférieures à l'unité. Désignons la plus grande de ces quantités par η et désignons par M la valeur $\max \frac{q}{(\varphi_2 - \varphi_1) R_2^2}$. Si l'on a $\max V_{n-1}(R_2, \varphi) \leqslant Q$ sur I et sur Δ, on aura

$$\max V_n(R_2, \varphi) \leqslant \frac{Q \int_I R_1^2 d\varphi}{\int_I R_2^2 d\varphi}$$

ou

$$\max V_n(R_2, \varphi) \leqslant \frac{Q \int_\Delta R_1^2 d\varphi}{\int_\Delta R_2^2 d\varphi} \text{ sur } \Delta$$

et $\max V_n(R_2, \varphi) \leqslant \eta Q$; par suite, $\max V_n(R_2, \varphi) \leqslant \eta^n M$. Par conséquence la série $\sum_{n=1}^{\infty} V_n(R, \varphi)$ converge dans le domaine fermé $G_1 + G_2$ vers une fonction harmonique V.

Démontrons maintenant que V satisfait aux conditions posées. En effet,

$$N(\varphi) = R_2^2 V(R_2, \varphi) - R_1^2 V(R_1, \varphi) = R_2^2 V_0(R_2, \varphi) + \\ + \sum_{n=1}^{\infty} R_2^2 V_n(R_2, \varphi) - R_1^2 V_{n-1}(R_1, \varphi).$$

En vertu des conditions (2) toutes les intégrales qui suivent sont nulles:

$$\int_I [R_2^2 V_n(R_2, \varphi) - R_1^2 V_{n-1}(R_1, \varphi)] \, d\varphi = 0,$$

$$\int_\Delta [R_2^2 V_n(R_2, \varphi) - R_1^2 V_{n-1}(R_1, \varphi)] \, d\varphi = 0,$$

$$\int_I R_2^2 V_0(R_2, \varphi) \, d\varphi = 0,$$

et nous avons

$$\int_\Delta N(\varphi) \, d\varphi = \int_\Delta R_2^2 V_0(R_2, \varphi) \, d\varphi = q.$$

Et de la même manière

$$\int N(\varphi) \, d\varphi = 0.$$

Donc, la fonction $N(\varphi)$ satisfait aux conditions (1) et notre théorème est démontré. Il en suit que pour les domaines convexes et même pour les domaines étoilés par rapport à l'origine, le problème possède une solution unique.

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PHYSICS

PROPAGATION OF SUPER-SONIC WAVES IN LIQUIDS

By B. G. ŠPAKOVSKIJ

(Communicated by S. I. Wawilow, Member of the Academy, 4. XII. 1937)

1. DISPERSION IN SOME PURE LIQUIDS AND AQUEOUS SOLUTIONS

The problem of dispersion of super-sonic waves is one of the most significant in molecular acoustics. When we have solved this problem it becomes possible, starting from the established dependence of elastic wave propagation velocity upon frequency, to integrate our conception of the structure of liquids in respect of their thermodynamical and molecular kinetic properties. The determination of dispersion of super-sonic waves in liquids should secure, as it did in the theory of gases, additional evidence concerning the nature of this complex aggregate state, the structural details of which are being examined by physicists from all sides. Numerous attempts to establish the presence of dispersion of super-sonic waves in liquids have not yet led to any direct positive results, apart from the brief communications of Hiedemann, Seifen, Schreuer ([1]) and Raghavendra Rao ([2]).

The purpose of this paper is to present a preliminary communication of experimental results of measurements of the velocity of propagation of super-sonic waves in some pure liquids and solutions taken within a wide range of frequencies.

The experimental technique, based on the method of the ultra-acoustic interferometer, has already been described, in its main aspects, by the author ([3]). The introduction of several improvements made it possible to obtain stably recurrent values of velocities of super-sonic waves, varying within the limits of $1—2^0/_{00}$. The number of standing half-waves, produced in the liquid on moving the reflector of the ultra-acoustic interferometer, varied from several tenths, at low frequencies, to several hundreds, at high frequencies; consequently, the length of the super-sonic wave could be determined with great accuracy. The frequency of the generated vibrations was determined by the aid of two precision wave-meters (for each range) with corrections, included in their calibration tables, computed on the basis of comparison with several piezo-quartz plates certified as standards of definite frequencies. The inconstancy of temperature conditions could be a source of considerable error as each procedure took considerable time, during which the change in the temperature of the examined liquid might amount to several tenths of a degree. Since the value of the temperature coefficient of the velocity of propagation of ultra-sound is considerable, it was necessary to introduce a correction in the final results. The value of this correction was determined graphically, by plotting the results of measurements against

temperatures and drawing a line of this function. Three plates of piezo-quartz: $40 \times 40 \times 8.0$ mm^3, $40 \times 40 \times 0.7$ mm^3, cut perpendicularly to electric axis (Curie cut), and a plate 1.0 mm thick, cut after Straubel, were used to generate super-sonic waves in liquids. On account of the flatness of the resonance curve set into vibrations along the thickness of the piezo-quartz plate, lubricated with vaseline oil to insure a better acoustic contact, and generating super-sonic waves in the liquid, each of the mentioned plates could be excited not only at resonance frequencies, but within a comparatively wide range of frequencies, and generate super-sonic waves in the liquid of sufficient power for the action of the ultra-acoustic interferometer. An essential improvement of the experimentation method was in that the velocity of propagation of super-sonic waves was mesured at several close frequencies, within the limits of each range, in order to avoid a possible systematic error due to phenomena of diffraction, as well as to the lack of uniformity of the super-sonic field. It was found possible, after preliminary computations, to disregard the correction of the velocity of propagation, due to inner friction, even when considering the structure of the super-sonic field as consisting of separate, rather narrow clusters of super-sonic waves propagated in an undisturbed liquid. The influence of the material and walls of the vessel, containing the investigated liquid, upon the value of the velocity of propagation of super-sonic waves, was not taken into account, since, according to the detailed investigations of Hubbard and Loomis (4) one may neglect this influence, because of the relations existing between the lengths of supersonic waves and the size of the radiating piezo-quartz plate and vessel. The possible error due to the systematic irregularity in the thread of the micrometric screw, could exert an influence only on the absolute values of the resulting velocities of propagation, but not change the comparative relationship between the lengths of ultra-sound waves of different ranges, and consequently it would likewise have no effect upon the relative values of velocities of propagation at different frequencies, calculated by means of these relations; these relative values being the only necessary ones in solving the problem of the existence of dispersion. Possible errors due to irregularities in the action of the micrometric screw were partly compensated by the large number of measurements taken, and their influence on the final result was taken into account together with other errors by means of the method of least squares.

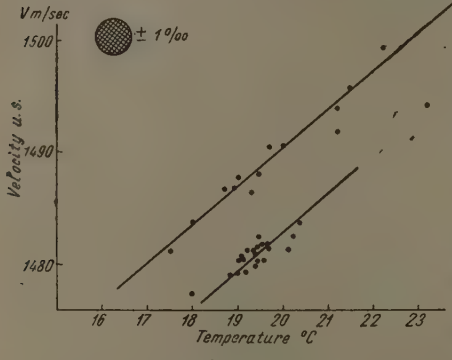

Fig. 1.

An absence of a marked dispersion in a number of liquids was observed, when making preliminary measurements of the velocity of propagation of super-sonic waves within a wide range of frequencies; namely in amyl acetate, ethyl acetate, ethyl bromide and some others.

The measurements of the velocity of propagation of ultra-sound waves in ethyl iodide were taken with especial care since, because of reasons previously mentioned by the author, a marked dispersion might be expected in this medium, due to the mechanism established by Kneser for polyatomic gases. As the value of the vibration quantum of a molecule of ethyl bromide is relatively small, the falling out of vibrational degrees of freedom at

very fast, in comparison with the time of relaxation, processes, should reduce in a marked degree the active value of specific heat, and, consequently, considerably raise the velocity of propagation of ultra-sound. 14 measurements of the velocity of propagation of ultra-sound were taken, within the range of frequencies of 200 to 300 kHz, on the basis of which a mean value of 889.7 ± 1.4 m/sec. reduced to 16° C, was obtained by the method of least squares. 11 measurements were made within the range of 3800 to 4000 kHz and a mean value of 888.6 ± 0.5 m/sec. was obtained in the same way. By comparing the resulting values of the velocities of propagation, differing from each other within the limits of measurement errors, one is warranted to infer the absence of marked dispersion within the limits of the range examined.

The results of measurements of the velocity of propagation of super-sonic waves in distilled water are given in Fig. 1, the upper line passing through

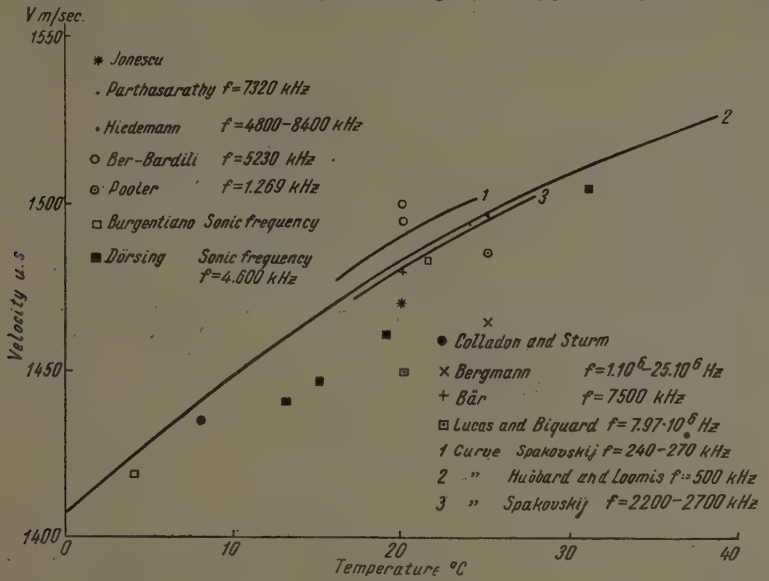

Fig. 2.

the points, obtained at a low range of 240 to 270 kHz for 14 different frequencies, and the lower line passing through the points obtained at a high range of 2200 to 2700 kHz for 18 different frequencies. The frequency of the high range was determined by means of the precision wave-meter and by comparing it with the övertones of the Pierce piezo-electric oscillator, the latter being stabilized with 3 different standards of quartz-plates. The low range frequency was measured by means of the precision wave-meter of guaranteed precision of $1^0/_{00}$ and checked before, as well as after, the measuring by comparing it with the overtones of the Pierce piezo-electric oscillator.

By examining the data of Fig. 1 we infer the presence of a slight dispersion of approximately $5^0/_{00}$, lying beyond the limits of experimental error and having a sign the reverse of that, determined by Kneser for polyatomic gases. The results of measurements of the velocity of super-sonic waves in distilled water, performed by different authors, at various frequencies and temperatures and by means of different methods (5), are compared in Fig. 2.

By examining the data in Fig. 2, we find that the measurements carried out by different investigators, even by means of nominally identical methods,

do not give the same results, consequently, by examining the data on the whole one may not draw the conclusion of a definite relationship between the velocity of propagation and the frequency. Moreover, the measurement of the velocity of propagation of sound-waves, may not be regarded as having the same significance, as the measurements of the velocity of propagation of super-sonic waves, on account of the uncertain value of the correction due to the elasticity of the walls of the vessel, which becomes noticeable on comparing the length of propagated waves to the size of the vessel.

The study of the relation between the velocity of propagation of elastic waves in electrolytes and the frequency, is of particular significance in the theory of electrolytes and in some problems of solvation. With this purpose in view, some preliminary measurements of the velocity of propagation of super-sonic waves in saturated aqueous solutions of potassium bromide and iodide were taken. The velocity of propagation in KBr solution was measured within a high range of 2800 to 3900 kHz at 3 different frequencies and a value of 1568.0 m/sec.$\pm$1.5 m/sec., at 18° C, was obtained by the method of least squares. Within low ranges of 270 to 290 kHz the velocity of propagation of ultra-sound at 3 different frequencies was 1578.3$\pm$1.7 m/sec. at 18° C.

In a saturated solution of Kl, the velocity of propagation of ultra-sound waves within high range of 3860 to 3910 kHz at 4 different frequencies and the velocity within low range of 269 to 275 kHz at 3 different frequencies, was determined.

The resulting values, determined by the method of least squares and reduced to a temperature of 16° C, were 1491.3 m/sec.$\pm$2.0 m/sec. and 1501.2 m/sec.$\pm$1.0 m/sec. for the high and low range respectively.

On examining the given figures we may conclude that the velocity of propagation of super-sonic waves in saturated solutions of KBr and KI is diminished by increasing the frequency.

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PHYSICS

PROPAGATION OF SUPER-SONIC WAVES IN LIQUIDS

By B. G. ŠPAKOVSKIJ

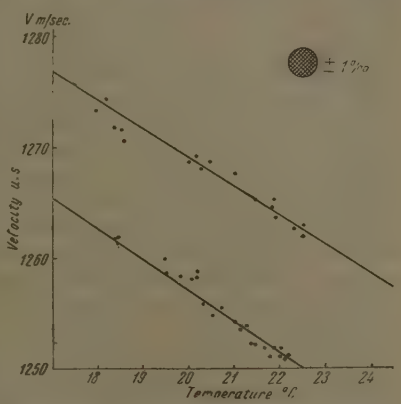
(Communicated by S. I. Wawilow, Member of the Academy, 4. XII. 1937)

2. DISPERSION IN ACETIC ACID

On making preliminary measurements of the velocity of propagation of super-sonic waves in different liquids, carried out by means of the ultra-acoustic interferometer method, the author observed a marked decrease of the reaction on the vibrating plate of piezo-quartz in acetic acid when the reflector was moving off even at comparatively low frequencies of the order of 300 kHz, while no reaction whatever was observed at frequencies ten times higher. This property of acetic acid, which distinguishes it in a marked way from other liquids, similar to it in other respects, in their viscosity coefficient in particular, should be attributed to its enormous absorption of super-sonic waves. This assumption was experimentally confirmed later on by Bažulin (1).

The purpose of the present paper is to present the results of measurements of the velocity of propagation of super-sonic waves, carried out in acetic acid within a wide range of frequencies by the method of the ultra-acoustic interferometer. The acetic acid, made use of, was designated as technical, contained some water, and was distinguished by a marked yellow tinge.

Some modifications were introduced in the experimental technique, previously described by the author (2).



Ultra-sound spreading velocity in acetic acid on a low scale (lower straight line) and on a high scale (upper straight line)

As a possible systematic error in the calibration of the precision wavemeters might distort the determination of the frequency of generated vibrations, a Pierce stabilized piezo-electric oscillator was used, loosely coupled with the high frequency vibrations generator which by means of inductive coupling produced elastic vibrations in the plate of piezo-quartz. A telephone, included in the anode circuit of the Pierce piezo-electric oscillator, gave the opportunity to hear a very large number of beats, arising at slow changes in the capacity of the oscillation circuit of the generator when some n-overtone of the

latter coincides with an m-overtone of the Pierce piezo-electric oscillator.

On measuring with the wave-meter the frequencies of vibrations, at which the so-called «zero beats» are heard, one may determine exactly the order of the coinciding overtones n and m (where n and m are integers) and, consequently, determine the frequency of vibrations, in relation to the stabilizing piezo-quartz. Making use of the same stabilizer (at 749.7 kHz) both within low (240—270 kHz) and within high (2400—3400 kHz) ranges, one may be sure of the correct determination of the relative values of frequencies, measured with absolute precision, up to the frequency of the stabilizing piezo-quartz. The measurements of the propagation velocity by the aid of every quartz-plate, were carried out within one range at several close frequencies, in order to avoid errors which could result from the lack of uniformity of the super-sonic field. Such lack of uniformity due to the complicated form of the vibrations of the piezo-quartz plate might be expected on the basis of investigations performed by Rschevkin ([3]) and Bär ([4]). The improvements introduced in the experimentation method, described above, considerably increased the reliability and accuracy of the measurements.

Within a low range the value of propagation velocity at 25 different frequencies in the interval of 220 to 270 kHz, was determined by means of a piezo-quartz plate $40.0 \times 40.0 \times 8.0$ mm^3 in size, cut perpendicularly to the electric axis (Curie cut). Within a high range, the propagation velocity was measured at 16 different frequencies in the interval of 2400 to 3400 kHz, with the aid of a plate of piezo-quartz 1.0 mm thick, cut by the method of Straubel. The results of the measurements are represented in the graph as functions of the temperature. Straight lines drawn through the points corresponding to the lower and upper ranges represent the change of the temperature function and permit of introducing temperature corrections and reducing the measurements to one temperature. The data obtained warrant us to conclude that the propagation velocity of super-sonic waves at high frequencies exceeds that at low frequencies by $1^0/_{00}$, the increase considerably exceeding the degree of scattering of points; this is accounted for by the errors of measurements within the limits of $1—2^0/_{00}$. As to sign and order of its magnitude the dispersion obtained resembles that, established by Kneser, on measuring the velocity of propagation of super-sonic waves in polyatomic gases. Liquids and gases resembling in many respects, it may be presumed to suppose that the cause of the observed dispersion of super-sonic waves in liquids should be attributed to the same mechanism as in gases. Theoretical considerations worked out in a very general way by L. I. Mandelstam and M. A. Leontovich ([6]), bearing on the absorption of sound in liquids, lead to the conclusion that a dispersion sets in when the period of propagated super-sonic waves is comparable with the value proportional to the time of equilibrium becoming established: $\tau' = \tau \frac{C}{c}$, where τ is the time of setting at equilibrium; C and c, the specific heat at the balanced and unbalanced processes respectively. Using formulas (12) and (13) of the above-mentioned article ([6]), we obtain the following expression connecting the value τ' with values that may be directly measured:

$$\tau' = \frac{1}{\alpha V_0} \frac{V_\omega - V_0}{V_\omega},$$

where V_0 and V_ω represented the propagation velocity of super-sonic waves at infinitesimal cyclic frequencies and at the cyclic frequency ω respectively and α is the coefficient of absorption of super sound at the cyclic frequency ω. The above formula may be applied to the preliminary determination of value τ' if use have been made of super-sonic wave

propagation velocity within high and low ranges. The value in the formula of the coefficient of absorption of super-sound at the cyclic frequency ω corresponding to the average cyclic frequency within high ranges, namely 2π 2800 sec.$^{-1}$ is not exactly known and may be determined indirectly by the extrapolation of the experimental data of Bažulin [1] who measured the coefficient of absorption of super-sound at different frequencies in pure acetic acid. On performing the extrapolation we obtain the value 0.37 for the absorption coefficient at the frequency 2800 kHz. Although the additional small quantities of water and other ingredients may have a marked influence on the value of the coefficient of absorption in acetic acid, the order of the value of the time of setting at equilibrium might not be influenced in a marked way. On examining the Bažulin's [1] experimental data on the change of absorption coefficient with frequency, under the aspect of the theory of Mandelstam and Leontovich [6], one may infer that the linear change of the coefficient of absorption, observed at frequencies of the order of 10^7 Hz indicates that at those frequencies the stage is attained at which dispersion should already be observed. Consequently, the determined experimental values of the velocity of propagation within high ranges, may be considered as value V_∞, and within low ranges, as value V_0. Substituting the numerical values obtaned in this way in the above formula, we obtain the preliminary value 10^{-7} sec. of the time of setting at equilibrium. This value may not be given a one-valued physical interpretation, on account of the absence of a sufficiently detailed molecular theory of the structure of liquids. One may suppose that the value τ and the proportional value τ' determine the lifetime of the excited molecule of acetic acid, in the same way as has been assumed by Kneser [5] in explaining the established dispersion of super-sound in carbon dioxide. However, such an explanation of the dispersion of super-sound in acetic acid is hardly plausible, since, according to the definitions of Daure the values of the vibration quanta of the complex molecule of acetic acid are comparatively large. In this way, their possible falling out at high frequencies, should not, as has been demonstrated by the author, produce a marked influence upon the value of the velocity of propagation. It is certainly more likely to look for the cause of dispersion starting from the theory of molecular associations. Smyth and Rogers [7] on the basis of their own investigations as well as of other authors, deduce the existence of simple and double molecules of acetic acid in a condition of chemical equilibrium in solution of acetic acid, and assert on the basis of all known facts, that the general character of behaviour of acetic acid depends on the complicated interaction of single and double molecules. From the standpoint of this hypothesis, one may consider the value τ, the preliminary determination of which is given above, as determining the time during that the equilibrium between single and double molecules of acetic acid is becoming established.

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PHYSICS

INFLUENCE OF MAGNETIC FIELD ON THE POTENTIAL GRADIENT IN PLASMA

By E. REICHRUDEL and G. SPIVAK

(Communicated by S. I. Wawilow, Member of the Academy, 8. XII. 1937)

The potential gradient in a positive column of gaseous discharge represents a fundamental parameter, and has been intensely studied for some time (1).

One of the most convenient methods of studying elementary processes occurring in plasma is that of exerting external influence upon the discharge (2). The authors have suggested a new quantitative method of plasma study using the disturbance of discharge by an external magnetic field (3).

In the course of this study it proved necessary to find out the influence of magnetic field upon the longitudinal gradient of the potential.

The influence of magnetic field on a stationary discharge has already been mentioned in a series of previous papers (4); however, these studies were either purely descriptive, or performed under insufficiently controlled experimental conditions. Later on (5) the influence of magnetic field on the potential difference pertaining to the whole discharge tube was examined. As the influence of magnetic field changes according to the region of the discharge, no satisfactory results can be obtained without examining the effects observed at separate regions of the discharge. Laws which are most typical for the discharge should be expected in the plasma. In a paper dealing with the coefficient of magnetic permeability of plasma (6) small change of gradient with magnetic field is mentioned, but no curves are added. As it may be seen below, our data prove that the change of gradient should not always be considered small whatever pressure, discharge current, etc. The changes of gradient reach sometimes 50—60 per cent. There are regions, however, where the gradient changes in magnetic field are insignificant. We have examined the influence of uniform longitudinal field in mercury vapor and rare gases (Ar, Ne).

Gradient measurements in rare gases were performed in a cylindrical tube supplied with an oxyde cathode (Fig. 1). The length of the tube was about one metre, its diameter being equal to 4 cm. Cylindrical or spherical platinum collectors were introduced into the tube 7—8 cm apart. The collectors were fixed with solder perpendicularly to the axis of the tube so that their middles were situated on the axis. The magnetic field was obtained by means of a long solenoid which was put round the tube. In order to prevent any influence upon the remaining parts of the discharge, the solenoid was given a length smaller than that of the positive column. Gradient measurements

in mercury vapor were made in a lateral tube added to the bulb (Fig. 2). This tube was of about the same s ape and dimensions as the tube shown in Fig. 1. The gradient was determined by the method of potential difference compensation between an arbitrary pair of probes (*1*), while the discharge current was maintained constant. The change of the gradient with magnetic field is plotted in Fig. 1 (for argon) and in Fig. 2 (for mercury). The curves are derived from numerous measurements, thus, they may be considered typical. The discharge current is the parameter of curves in Fig. 1, while the cathode temperature is the parameter of curves in Fig. 2. In absense as well as in presence of magnetic field there must be established a gradient value in the plasma which would ensure an adequate conductivity. As to the changes of energy losses caused in the discharge by the magnetic field, they are compensated by corresponding changes of gradient. The magnetic field may cause the losses to change both in the gas volume and on the walls of the discharge tube. The magnetic field

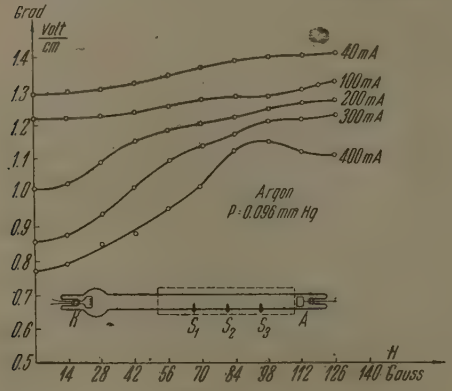

Fig. 1.

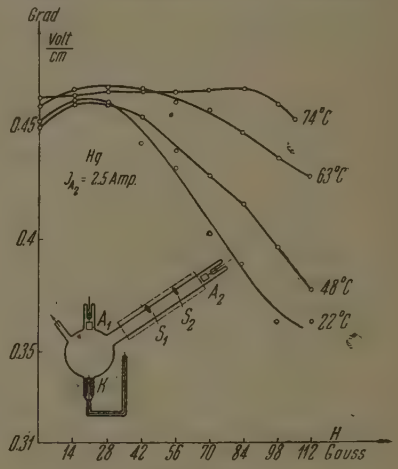

Fig. 2.

tells on the motion of electrons in plasma. An increase of losses due to elastic collisions with gas atoms may be expected at the expense of the apparent increase of pressure ([7]); on the other hand, a marked decrease of the coefficient of electron diffusion towards the walls of the discharge tube is observed ([8]). If the former factor must result in an increase of the gradient, the latter may act in the opposite sense. Moreover, the magnetic field may markedly change the unelastical energy losses of electrons occurring in the discharge. One may attempt to explain the run of curves in Fig. 1 and 2 by a play of opposed effects of the magnetic field. It will be possible to say to what extent the effects which are supposed to account for the run of the gradient are fundamental not before a quantitative theory of plasma in the magnetic field is developed. It is known that at moderate low pressures Shottky's theory ([9]) expresses fairly well the typical properties of plasma, connecting the longitudinal gradient with the discharge parameters. Taking a more accurate account of the electron energy losses (not only of those caused by gas ionization, but also of other unelastical losses), and of the change of the coefficient of diffusion, proceeding in a direction perpendicular to the magnetic field, one may attempt to generalize Shottky's theory, and to explain the run of the gradient observed by us. In one paper ([10]) we found the assumption that the influence of magnetic field on the plasma cannot be explained on the basis of Shottky's theory, since

the plasma becomes unhomogeneous lengthwise. The experiments referred to in this paper have been made on Lübcke's wall-amplifier ([11]) (Wandstromverstärker), containing considerable ferromagnetic parts. The authors themselves admit that the presence of ferromagnetic parts might have essentially distorted the data obtained by them ([10]). We have tested plasma for uniformity in the magnetic field, determining the gradient at various regions of plasma. In all cases the same values were obtained.

A theoretical interpretation of this phenomenon is moreover complicated by the fact that the change of discharge current is accompanied by a change of gas density across the tube.

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PHYSICS

THE DISINTEGRATION OF NUCLEI BY COSMIC RAYS

By A. FILIPPOV, A. GDANOW and I. GUREVICH

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In the course of the last two years we investigated the disintegration of nuclei under the action of cosmic rays. As indicators photographic plates with a special emulsion (E_3) about 50μ in thickness worked out by one of us (1) were employed. The experiments were carried out on the sea-level. A set of plates was impregnated with borax and lithium compounds. The exposure lasted 200—800 h. A study of

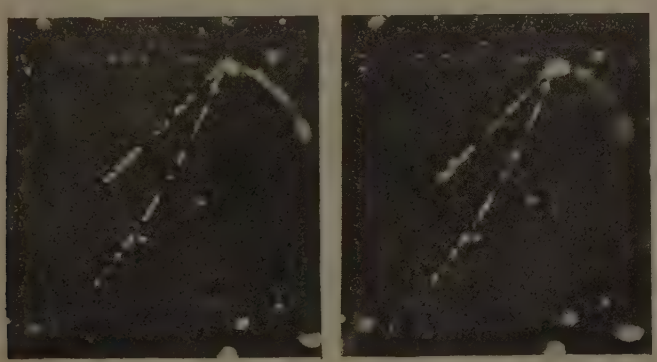

Fig. 1 (×370).

the plates under the stereomicroscope has revealed a large number (of the order of 10^4) of disintegrations with the emission of 1, 2, 3, 4 and 5 particles. Single tracks are to be attributed to the disintegration of nuclei, since their direction is not connected with the direction of the primary cosmic radiation. The greatest length of single tracks was found to be 304 cm of air (15°C, 760 mm) or 2130 μ in emulsion. If we attribute the longest track to proton, its kinetic energy must be ~17 MeV. However, the density of the grains in the emulsion at the end of some of the tracks is higher than that for protons; this shows that some tracks are due to more heavy particles. If we attribute the longest track to an α-particle, the maximum energy of single tracks will be 68 MeV. Some of the single tracks clearly belong to α-particles and are as long as 80 cm and even more.

Four cases of disintegration into three and four particles are seen on Figs. 1, 2 and 3. Fig. 1 shows the disintegration into four particles, the centre of the disintegration being situated in the emulsion itself (two long tracks strike the glass). The lengths of these tracks are 8, 20, 10 and 12 cm of air; their curvature may be explained by some deformation of the gelatine layer. Measurements of distances between grains show that the second and the fourth tracks from the right-hand side belong to single-charged particles, the third one—to an α-particle and the first one—to a particle with a charge larger than 2. If we ascribe them to disintegration of N^{14} according to the following scheme: $N^{14} + E = Li^4 + H^3 + He^4 + H^1$, the kinetic energies of reaction products will make 14.8, 5.7, 9.6, and 2.8 MeV respectively. If we sum up their values adding $\Delta M = 35$ MeV, we will obtain $E = 68$ MeV. This value of E is no doubt underestimated, since not all tracks are lying

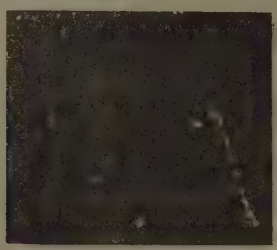

Fig. 2 (×200).

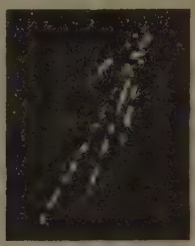

Fig. 3 (×430).

completely in the emulsion. It is of interest to note that other possible kinds of N^{14} disintegration will not change E very much. A study of the balance of energy and impulse shows that if we assume that at the disintegration non-ionizing particles are not generated, the reaction cannot be attributed to the disintegration of the nucleus by a cosmic particle, accompanied by its full absorption. It seems natural to suppose, that a fast cosmic particle, at the passage through the nucleus, gives up to it some of its energy, stimulating a disintegration(2) (a «warming» of the nucleus and the subsequent «evaporation» of the particles). It is possible, however, to attribute the tracks diverging from one point to the disintegration of heavier nuclei (Ag or Br), the residue of which though does not give a visible track, yet receives a part of the impulse.

The tracks shown on Fig. 2 (right) demonstrate a disintegration into three particles, their paths are 2.8, 2.3 and 11.4 cm of air. The both short components may be ascribed to α-particles, and the longer one, to a single-charged particle. It is yet difficult to make a definite statement about the nature of the disintegrating nucleus, since the emulsion contains nuclei of H, C, N, O, Ag and Br. If we attribute these tracks to the disintegration of B^{11} (the plate was impregnated with borax), the full energy of reaction will be of the order of ~ 23 MeV. Figs. 3 and 2 (left) show groups of heavy particles with approximately the same direction—out of the glass. Their paths are 4.1, 11, 9.4 and 7 cm of air (Fig. 3) and 7.4, 17.8 and 4.4 cm of air (Fig. 2) respectively. The particles of Fig. 3, as may be judged from the average distance between the grains, are protons (proton shower). The particles of the shower shown on Fig. 2 (at the left) present an especial interest since for the first and second components the distance between the grains is too large to be ascribed to slow protons. The third

component belongs to a proton. The average distance between the grains of the first track is 5.1 μ while the proton with the same track would give 1.6 μ. It seems natural to ascribe them to a feebly (in comparison with the proton) ionizing particle. On the basis of the average distance between the grains it is possible to evaluate (very roughly) the mass of such a semi-heavy particle assuming that its charge is equal to 1. For one track (the extreme left track, Fig. 2) such an evaluation shows that mass μ of the semi-heavy particle does not exceed 180 m_e. For several other tracks (one of them is the central track of the shower, Fig. 2) $\mu \leqslant 800\, m_e$. It is possible that these particles observed by us are connected with the semi-heavy particles of Neddermeyer and Anderson [3].

Blau and Wambacher [4] have recently published a note in which they communicate their observations made on the height of 2300 m after the same method. They describe tracks diverging from one point («stars»), that belong probably to the same class of phenomena as the disintegrations observed by us. The disintegration into three particles under the action of cosmic rays was observed earlier in a Wilson chamber by Anderson and Neddermeyer, and Brode and others [5].

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PHYSICAL CHEMISTRY

MECHANICAL PROPERTIES (VISCOSITY, RIGIDITY AND STRENGTH) OF ADSORPTION LAYERS AND THEIR RELATION TO THE AREA PER MOLECULE AND NATURE OF UNDERLYING LIQUID

By A. A. TRAPEZNIKOV and P. REHBINDER, Corresponding Member of the Academy

In the studies conducted since 1934 (1) the authors have succeeded in quantitatively comparing the mechanical properties of adsorption layers formed by surface-active substances, insoluble in water (higher fatty acids, alcohols and esters), for different values of area per molecule on water or aqueous solutions of electrolytes.

Notwithstanding the numerous and detailed studies on the equation of state of monolayers [Langmuir (2), Adam (3), Rideal (4)] the attempts of the latter authors to determine the state of the layers according to their mechanical properties, were only based upon qualitative experiments on the motion of dust particles, placed on the film, under the action of an air jet. The state of a film was considered liquid when the particles slid along the surface, and solid when they remained almost stationary. For most substances investigated by Langmuir (2), the points on the two-dimensional pressure curves $F=f(a)$ corresponding to the area a at which the films become solid, are marked with the symbol S. Adam points out that the films of various substances possess different mechanical properties, but fails to characterize them quantitatively. Mouquin and Rideal (5) determined the modulus of rigidity of saturated monolayers of palmitic and stearic acids, and found it to be $4.9 \cdot 10^7$ dynes/cm^2; according to the latest papers by Langmuir (6), Myers and Harkins (7) on the effects of polyvalent cations present in the liquid underlying the film, and also according to our own studies of the influence of cations on the mechanical properties of films, this is due to the formation of a soap film, i. e. to the reaction of the acid film with polyvalent cations.

We used the combined method of Coulomb, Schütt (8) and Rhode, which consists in measuring the logarithmic decrement, λ, and the period of oscillations, T, of a disc with a given moment of inertia, the disc being suspended on a thin torsion wire and placed in a monolayer which was compressed by a barrier sliding along a paraffine trough filled with water. When the films exhibited solid or plastic properties, the angular displacement of the disc was measured upon twisting the upper end of the same wire. This method allowed us to carry out a quantitative investigation of the relation between the mechanical properties of the films of numerous substances and the area per molecule along the F—a curves.

We substantiated the general proposition, expressed long ago by P. Rehbinder (9).according to which the surface viscosity or surface strength (e. g. resistance to shear) in adsorption layers increased with the surface concentration and reaches a maximum upon their complete saturation. The existence of a maximum of mechanical strenght in non-saturated layers, reported by Talmud (10), was not borne out by our experiments. It was further ascertained that the resistance to shear (in the main viscous and elastic) in monomolecular films is developed continuously according as the film is compressed and exhibits a particularly sharp rise on transition to the condensed state. A further compression of a saturated monolayer in cetyl alcohol and arachidic acid films results in a decrease of viscosity, which fact may apparently be accounted for by the formation of polymolecular nuclei—microlenses. The isotherm of resistance to shear has a form, similar to that of the $F—a$ isotherm. The compression of films involved a sharp rise of the resistance to shear for palmitic acid (in presence of traces of cations, e. g. $Al^{\cdot\cdot\cdot}$) at 24 Å^2, for

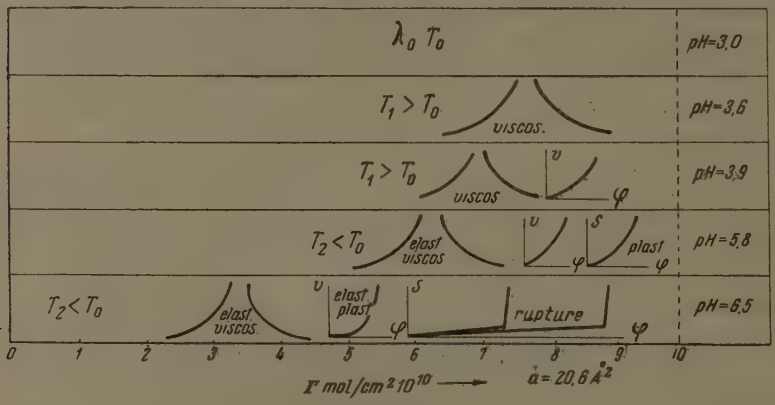

Fig. 1.

tripalmitine at 65 Å^2 and for cetyl alcohol at 20.5 Å^2. As regards qualitative changes in the mechanical properties, these were not observed over wide ranges of areas per molecule, i. e. definite «fusion points» were observed in the films. After our investigations had been completed, Myers and Harkins(11) who had used a similar method published a paper in which a break on the curve $F—a$ was reported, this break corresponding to the intersection of two straight lines, one of which belongs to the liquid film, and the other corresponds to the plastic (solid) one in the condensed region. It follows from our graph given in Fig. 1, that this transition (from the liquid to plastic-solid state), which corresponds to pH=3.9, is not characteristic of the break on the $F—a$ curve, since some films are viscous until their collapse (at pH=3.0), whilst others are elastic, i. e. solid (at pH=6.5), in the region of areas per molecule, much larger than that corresponding to the break on the $F—a$ curve. The sharp rise of resistance to shear on complete condensation involves no qualitative transition from the liquid to the solid state.

We found, in agreement with the qualitative data of Langmuir (6), Myers and Harkins (7), that on pure water the palmitic acid film remains viscous up to the saturation point, whilst the presence of traces of polyvalent cations*

* The importance of this factor was kindly indicated to us by A. N. Frumkin, Member of the Academy, and V. A. Kargin in connection with the papers by Langmuir (6), Harkins and Myers (7), on the specific action of cations, which had just appeared.

results in the formation of elastic films whose tensile strength is 10—11 dynes per cm. In the latter case the acidulation of the underlying liquid gradually transfers the film from the elastic state through the plastic region to the viscous state, whilst, for each constant pH value, an increase in the cation concentration involves the solidification of the film. Identical mechanical properties may therefore be obtained by simultaneously varying the concentration of both cations and hydrogen ions. This phenomenon is associated with the formation of soap in the monolayer. When traces of cations are present, the liquefying action of the acidulation of the underlying

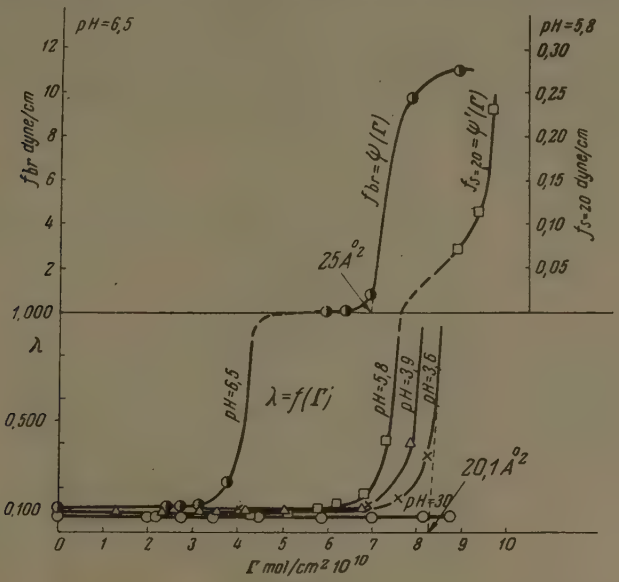

Fig. 2.

liquid upon fatty acid films is confined in substance to preventing the soap formation. For some films, especially in an expanded state, acidulation may produce a direct liquefying action. According to preliminary data, the intensity of the effect of cations, i. e. their ability to produce the solidification of layers, increases as the valency is increased in the series $Na^{+} < Ca^{++}, Cu^{++} < Al^{+++}$. The effect of cations, which seems to obey the law of mass action, is governed by a general scheme, but the pH region corresponding to the solid-liquid transition of films has a characteristic value for each individual cation.

Fig. 1 represents the graphs, obtained for palmitic acid, characterizing the variation of the mechanical properties of its films with the pH value of the underlying liquid and the area per molecule. For pH=3.0, the logarithmic decrement λ and the period of oscillations T within the limits of experimental error are indistinguishable from the corresponding values obtained on clean water (λ_0 and T_0). On further transition to higher pH values and on sufficient saturation of the layer, the viscosity becomes measurable, undergoes a further increase (rise of λ and T), structural viscosity becomes manifest, and the elastico-viscous type of motion, characterized by a decrease in T ($T_2 < T_0$), sets in. Near the saturation point the motion of the disc becomes aperiodic—untwisting of the wire with a velocity, φ, depending on the angle of twist, φ. For pH=6.5, a clearly pronounced tensile strength is to be observed in layers approaching the condensed state (the films become brittle

with hints of plasticity). This is revealed by S—φ curves which show the displacement on the optical lever scale as a function of the angle of twist. The lower part of Fig. 2 represents the experimental curves showing the variation of the logarithmic decrement for palmitic acid films on underlying liquids containing traces of multivalent cations, e. g. $Al^{\cdots}$, with the surface concentration, $\Gamma = \frac{1}{Na}$, of the layers for different pH values. In the upper part of Fig. 2 the left-hand curve shows the variation of the tensile strenght, f (in dynes/cm), as derived from the $S - \varphi$ curves; the right-hand curve represents the variation of the force, $f_{s=20}$, producing a definite arbitrary chosen strain corresponding to 20 divisions of the scale. The dotted lines connecting the λ—Γ curves correspond to saturation regions in which the motion of the disc becomes aperiodic. The left-hand curve in Fig. 3 shows the rise of the stress, $f_{s=20}$, characterizing the plastic tripalmitic films on different underlying

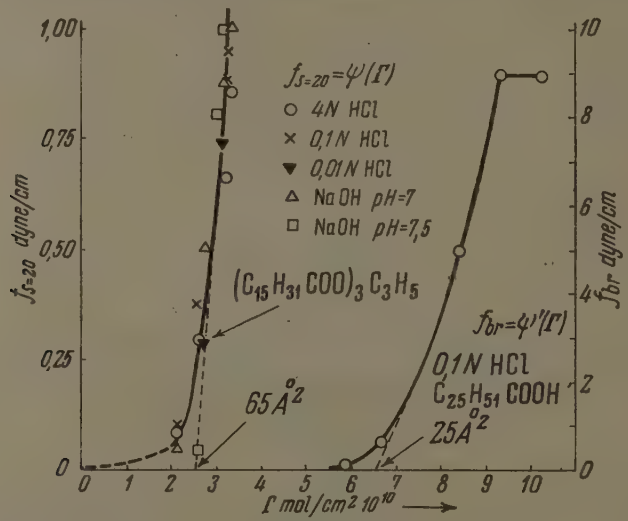

Fig. 3.

liquids, the $f_{s=20}=\psi(\Gamma)$ curve being independent of the pH values of these liquids; the right-hand curve illustrates the variation of the tensile strength of cerotic acid films on 0.1 *n* HCl.

An increase in the length of the hydrocarbon chain in homologous series resulted in a gradual enhancement of the resistance to shear on transition from viscous to plastic and, finally to elastic films; e. g., on 0.1 *n* HCl the palmitic acid film ($C_{15}H_{31}COOH$) proves to possess a low viscosity; the film of arachidic acid ($C_{19}H_{39}COOH$) is highly viscous; finally, that of cerotic acid ($C_{25}H_{51}COOH$) is elastic and possesses a tensile strength of the order of 9 dynes/cm (Fig. 3).

An interesting phenomenon, analogous to that of syneresis, was observed for cetyl alcohol films: on ageing of the condensed monolayer, the viscosity at first rapidly rises, reaches a maximum and falls down to the initial value. A repeated compression (after expansion) of both fresh and aged films involves a sharp increase in viscosity.

The fundamental factor affecting the state of a film is the length of the hydrocarbon chain, but the nature of the polar group also plays an important part; this is evidenced by the mechanical properties of cetyl alcohol and

ethyl palmitate films which behave like liquids possessing structural viscosity, whereas the viscosity of palmitic acid films is extremely low. Tripalmitine and cetyl stearate films are plastic. Oleic acid, trioleine and cholesterine films possess a very low viscosity lying within the limits of experimental error. However, on raising the sensitivity of the apparatus by using a thinner torsion wire and substituting a thin ring for the disc (as in our measurements of the viscosity of two-sided soap films—see our next communication in this series), quantitative data may also be obtained for the films of the latter class of substances.

The two-dimensional pressure curves (F—a) for the above films with strongly differing mechanical properties possess, according to Adam, nearly equal intervals corresponding to the condensed state. This shows that the method of measuring the resistance of unimolecular adsoption layers to shear may be used as an independent method of their study, these methods affording a good characterization of the peculiarities in the structure of the molecules in monolayers.

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GENETICS

THE COURSE OF THE DEVELOPMENT OF WHEAT EAR

By A. A. SAPEGHIN, Member of the Academy of Sciences of the Ukrainian SSR

In the phenogenetic study of the vegetation period of wheat I observed a quite distinct and direct connection between the number of spikelets in the spike and the interval from the beginning of the light-stage to the earing—in F_2 of about all the varieties crossed. This made me to study the concrete course of the light-phase before its hybridological analysis, planned at first, had been accomplished. Till now I have studied the development of the ear in 16 varieties and their 6 hybrids F_1.

This study brought to the establishment of characteristic phases of ear development common to all varieties and their hybrids examined by me.

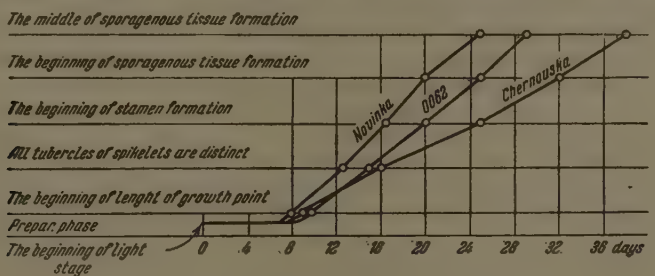

Fig. 1. Course of development of the wheat ear.

I am endeavouring to show these phases in the present communication, where they will be exemplified by three summer varieties: Novinka, Saratov-kind 0062 and hard wheat (Chernouska).

The investigation was the following. Seeds were sawn in pots and in order to secure a complete course of the yarovization left for 15 days under the action of a short day (8—9 hours) at spring temperatures (8—10°). Then the plants were simultaneously transported into a hot-house where they enjoyed a long day (16—18 hours) at a temperature 18—25°. The day they had been transported into the hot-house and afterwards, at short intervals (3—5 days), several tops of all the studied varieties were fixed in the Navashin mixture. Preparations of fixed top were made, which were not subject to change and thus afforded the possibility to observe the course of ear development under a microscope. This process is represented schematically in the figure.

On the horizontal are plotted the numbers of days from the beginning of the light-stage when the yarovized plants were exposed to a long warm

day. On the vertical at nominal distances are plotted morphologically well-defined moments of ear development up to the middle of sporogeneous tissue formation in the stamens of external spikelets at the middle third of the ear.

As is evident from the scheme and still more obvious from the preparations, the growth point of the stalk does not change its characteristic aspect during several days but remains short and forms leaf-tubercles. The latter are probably the subsequent leaves of the halm. In the conditions of the described experiment this phase, which could be called preparatory, lasted about 7 days for the Novinka, about 8 days for Chernouska and about 9 days for 0062. In another series of experiments the difference among these varieties was considerably greater: from 4—5 days for Avrora and Preliudka to 10—11 days for Cesium.

The preparative phase elapsed the development of the growth point changes abruptly. The latter is becoming longer and wider and at the same time is covering with spikelet tubercles. The phase of tubercle formation lasts from 3 to 7 days according to the variety. The number of tubercles formed during 24 hours varies from $2^1/_2$ to 4 in different varieties and depends upon environment, especially temperature. The number of spikelets in a spike depends upon the rate of tubercle formation and the duration of that phase. According to materials till now gathered by me, it most frequently occurs, that a long spikelet phase corresponds to a long preparatory phase. The rate of tubercle formations in this case is often comparatively somewhat reduced.

On the whole, long preparatory and spikelet phases lead in the main to a large number of spikelets. Hence the above-mentioned connection between the duration of the period from the beginning of the light stage to earing (in F_2) and the number of spikelets in a spike becomes clear.

The different times in which tubercles and then florets are formed cause the lines of ear development in different wheat varieties to diverge more and more, and the intervals between the posterior moments of development increase continuously. In the described experiment, e. g. the growth point of the Chernouska began to stretch a day later then that of the Novinka and the tubercle formation terminated 3 days later. Stamens begun to form 8 days later, while the sporegeneous tissue starts its formation 12 days later.

It is well known that the ear development in wheat accustomed to its environment requires a long day during 2—4 weeks, according to the variety. The figure shows that the end of this period, that have been called «light-stage» by T. A. Lysenko, rather exactly coincides with the sporogeneous tissue beginning to form, which gives finally (through the spores) the genital cells of wheat. Therefore one may surely suppose that the third stage of wheat development is the spore-hametogeneous sexual stage [see also the experiments of Kirichenko and Kraievoi (8)].

From the standpoint of morphological contents the yarovization stage is a vegetative one; the light-stage, a spore-leaf or spikelet one; the third, a sexual spore-hametogeneous stage and still there remains a fourth stage— that of the embryo and seed.

Literary data on the course of ear development are all fragmentary. Besides the fact that the development proceeds by stages [Philippchenko (4), Bell (1)] is not taken into consideration. The paper of Shestakov (5) is the only exception, but his data are too scrappy.

Of all the ear development phases the preparatory one is the most interesting. Here undoubtedly occur some changes in the structure of the growthpoint cells and accumulation of new elements takes place. This process having reached a certain quantitative minimum, the growth point changes its quality, quickly becomes larger and forms spikelet tubercles.

According to the works of Chailahian ([6]) and Moshkov ([7]), one can suppose that during the preparatory phase specific substances, indispensable for the development of reproductive organs and which are formed in leaves, must feed the growth point to a certain active minimum. But this supposition is not sufficient for one has to suppose that the leaves continue to produce these substances, while the number of the spikelets in the spike is limited. Therefore I am disposed to consider that the principal leading factor which determines the number of spikelets in the spike is concealed in the growth point while the special substance, which is flowing to from the leaves during the light stage and provokes ear development, is an accomodation to the altering environment. Perhaps the action of this substance is at last paralized under the influence of the spikelet tubercles themselves.

The question I had to solve was whether the preparatory phase is that «sensitive period» [Devries ([2]), Roelofs ([3]) and others] when the number of spikelets is being determined.

The investigating experiment which I was able to perform in 1937 confirmed this supposition. During the first 4 days of the preparatory phase I put one part of my plants into conditions of a weakened light, another part was secured with magnesium sulphate (1%), the third part remained for control. 4 days after the ground of all the plants was carefully washed out and since then they had a normal feeding, lighting and so on.

Result. The number of spikelets in a spike in the first two groups became less than in the control one (13 instead of 15); the grain was fine and weak, while in the group, which received a weakened light during the first 4 days, the earing was besides 5—6 days late and there was very little grain.

Of course, this experiment gives only a partly answer and now I am widely developing it in order to precize the phase limits of the ear development as well as to master its determinative moments. Besides I purpose to study the ear development under most various aspects on ecologically different wheat varieties, which have been established by the works of the All-Union Plant-Breeding Institute, and on varieties for which gen differences in the number of spikelets are known.

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PLANT PHYSIOLOGY

DROUGHT RESISTANCE AND DEHYDRATION RESISTANCE OF PLANTS

By G. N. EREMEJEV

(Communicated by A. A. Richter, Member of the Academy, 17. XI. 1937)

Our several years' study of the drought resistance of plants has resulted in a large amount of experimental data showing that the different resistance to wilting observed in the overground plant organs (leaves, branches) is, in the first place, due to their different abilities of retaining water in wilting. At the same time, only in very few papers on the subject ([1], [2], [3]) a distinction has been made between the resistance of plant tissues to wilting and their resistance to loss of water. In the overwhelming majority of papers on drought resistance of plants the resistance of plant tissues to wilting has been identified with their resistance to loss of water ([4], [5], [6], [7]).

In trying to determine the drought resistance of plants, it is very important to know whether the observed resistance to wilting is due to a greater ability of the tissues to endure dehydration or to other peculiarities.

Experimental data on the physiological nature of the resistance of plant tissues to wilting are very scanty, in spite of the fact that the resistance of plant tissues to wilting, without covering all the phenomenon of the drought resistance of plants, is, none the less, one of its essential aspects.

According to the available data ([1], [2], [3]), the resistance to wilting in various plants may be due both to the ability of the plant tissues to endure a deep dehydration and to their ability of retaining water in wilting. As shown by our data cited below, the leading factor of the resistance of the overground parts of fruit trees to wilting is their ability to retain water. The principal aim of our investigations was to devise methods of determining the drought resistance of fruit trees. We investigated the following plants: different varieties of almond-tree (*Amygdalus communis*), peach-tree (*Prunus persica*), olive-tree (*Olea europea*), pear-tree (*Pyrus communis*), and other fruit trees.

In the overwhelming majority of cases, only the overground organs are proper fruit tree varieties (grafts). Therefore, in our researches, the diagnosis of drought resistance consisted in ascertaining the resistance of the overground organs of plants to wilting by the following method: leaves or branches (identical, if possible) were cut from the plants under investigation, underwent wilting under identical conditions, and were then put into water in order that they might recover their turgor.

The extent to which they recovered their turgor was regarded as the principal criterion in estimating their resistance to wilting. Furthermore, in the

course of wilting, we determined the water-retaining power, the variations in the respiration intensity, and studied the behaviour of the stomata.

Table 1

The Water Content in the Leaves at the End of Wilting and the Power of Recovering Turgor after Wilting

Serial No	Kind of plant	Resistance of leaves to wilting	Duration of wilting in hours	Water content in the leaves, in percentages of dry weight		% of leaves that recovered their turgor after wilting
				At beginning of wilting	At end of wilting	
1	Peach-tree	Resistant	72	285—395	120—145	81—100
		Fairly resistant	72	285—300	79—109	50—73
		Non-resistant	72	270—295	45—75	9—20
2	Almond-tree	Resistant	24	175—200	85—115	79—100
		Fairly resistant	24	165—190	62—65	40—45
		Non-resistant	24	135—170	45—50	5—10
3	Pear-tree	Resistant	24	180	93	95
		Fairly resistant	24	200—210	53—69	43—50
		Non-resistant	24	170—180	42—48	3—10

The data of Table 1 show that the ability of the leaves to recover their turgor (and, together with it, other functions likewise) varied directly with the quantity of water retained in the leaves till the end of the experiment.

Table 2

Wilting Resistance and Water-Retaining Power of the Leaves of 5 Varieties of Almond-tree

Serial No	Variety	Amount of water given off by leaves during 24 hours of wilting in % of initial water content	Water content in leaves in % of dry weight		% of leaves that recovered their turgor	Degree of resistance to wilting according to decimal scale *	Agrobotanical estimate of drought resistance **
			At beginning of experiment	At end of experiment			
1	Crimean № 53 . . .	40	195	116	100	10	The most resistant
2	№ 50 . . .	50	200	80	75	7	Resistant
3	№ 174 . . .	64	168	62	61	5	Fairly resistant
4	№ 175 . . .	59	158	65	42	4	Fairly resistant
5	№ 44 . . .	70	154	47	5	1	Non-resistant
6	№ 180 . . .	80	137	28	9	1	Non-resistant

* Found after the method of wilting cuttings. Higher figure correspond to higher resistances.

** The agrobotanical estimate of the varieties was made by A. A. Richter, chief of the Section of Subtropical Cultures.

The data summarized in Table 2 show clearly what a prominent part is played by the water-retaining power in the resistance of leaves to wilting. It is characteristic of the data of Table 2 that the degree of drought resistance of the different varieties, as determined by the above-described method, coincides with that established as a result of prolonged field observations.

The drop in the transpiration intensity observed in the wilting leaves (Table 2) cannot be explained as only due to the transpiration being regulated by the «incipient drying up» of the cell walls (Livingston, 1912), the conditions of water supply ([5]) and the accumulation of assimilates in the leaves ([9]). But it was often the case that the leaves of the non-resistant varieties, having a smaller water reserve in their tissues (greater drying-up), gave the water off more freely than the leaves of the resistant varieties, whose tissues were saturated with water to a greater extent (slighter drying-up). Moreover, the lower leaves, which at the beginning of the experiment had had a smaller water reserve, none the less gave the water off more freely in wilting than the middle and the upper leaves (data of Table 3). The greater expenditure of water by the lower leaves of herbaceous plants during wilting has been pointed out in the literature ([10]). In our experiments, the lower water-retaining power of the lower leaves correlated with their lower resistance to wilting.

Table 3

Transpiration Intensity in the Wilting Leaves in Relation to their Position on the Shoots

Serial No	Kind of plant	Position of leaves on the shoot (from below)	Amount of water given off by leaves during 7 hours of wilting in % of initial water content	Water content in leaves at beginning of wilting in % of dry weight	Remarks
1	Apple-tree	Lower	71	176	The figures show the mean of 50 leaves in each variant
		Middle	62	192	
		Upper	32	200	
2	Almond-tree	Lower	22	255	
		Middle	15	270	
		Upper	10	282	
3	Peach-tree	Lower	15	249	
		Middle	14	289	
		Upper	11	308	

The fact that the leaves non-resistant to wilting expend more water during wilting, is accounted for by the transpiration being but poorly regulated by the stomata of these leaves. The importance of the responsiveness of the stomata to the action of drought for the plant's drought resistance has been recorded ([1], [8]). It has been shown ([1]), moreover, that the rye stomata lose their motor function under the action of drought. We have found out that the stomata of the lower leaves (relative to their position on the shoots) of fruit trees, under the conditions existing in Crimea, often lose their responsiveness to wilting and to being placed in the dark. This results in the fact that the lower and other leaves whose stomata do not function, give the water off more freely, under the same conditions of wilting, than do the leaves with normally functionning stomata.

The data on the loss of motor function by the stomata of the lower leaves during wilting and when deprived of light are shown in Table 4.

Table 4

The Response of the Stomata of Different Leaves to Wilting and to Being Placed in the Dark

Serial No	Kind of plant	Position of leaves	State of stomata before experiment	Description of experiment variants	State of stomata after experiment
1	Apple-tree	1—3	90% open	3-hour stay in the dark while on plant	80% open
		5—8	90% »		10% »
		2—3	95% »	3-hour wilting of cut leaves	85% »
		5—9	95% »		15% »
2	Peach-tree	2—3	95% »	3-hour stay in the dark while on plant	75% »
		5—9	95% »		5% »
		1—3	95% »	2-hour wilting of cut leaves	75% »
		5—9	95% »		5% »

Further investigations have shown the physiological activity of the fruit-tree leaves and the resistance of the latter to wilting to depend largely upon the extent to which the plants are loaded with fruit. For the cotton plant, the dependence of physiological processes upon fruit bearing has been pointed out in the literature ([11]). It appears from the experiments made by us that an increase in the relative «fruit load» (i. e. the leaving of a great number of fruits on the tree) caused a decrease of the catalase activity in the leaves. The resistance of the leaves to wilting likewise decreased with an increase of the «fruit load».

Our researches, briefly set forth above, warrant the following conclusions:

1) Differences in the resistance of the fruit-tree leaves to wilting, both in different varieties and within one and the same plant, are due, in the first place, to the ability of the leaves to retain water during wilting.

2) A low ability to retain water in wilting observed in non-resistant leaves is due to a weak response of the stomata of these leaves to wilting.

3) A temporary placing of the leaves in the dark during the daytime may be used (for plants closing their stomata in the night) as a method of determining the responsiveness of the stomata to external influences (drought, light).

4) A common feature of the leaves of all the fruit-tree varieties investigated was that the physiological activity of the leaves as well as their resistance to wilting decrease with an increase in the «fruit load».

5) The degree of the resistance of leaves and branches of fruit trees to wilting (other conditions being equal) is a reliable physiological index to be used in determining the drought resistance of a variety.

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PLANT PHYSIOLOGY

THE SIGNIFICANCE OF MINERAL ELEMENTS IN THE PROCESS OF YAROVIZATION

By G. ABOLINA

(Communicated by A. A. Richter, Member of the Academy, 9. XII. 1937)

The theory of yarovization, developed by T. D. Lysenko (1), which marks a new stage in the study of plant physiology, has produced, for almost the first time in the history of science, such remarkable practical results in a very short period of time and is indicating further means of gaining additional evidence on the physiology of development.

Yarovization is a problem of great importance and its close analysis is the urgent aim of present-day investigators.

Widely practicing the application of yarovization, it is essential to study those chemical and physiological changes, which accompany the transition of plants from one stage of development into another.

Proceeding from the theoretical thesis of T. D. Lysenko (2) that: «for plants to pass through the qualitative stage-transformations, a complex of relatively definite conditions of environment required by the vegetative organism at each stage, is necessary», we undertook the problem of determining the part played by separate elements of the mineral nutrition when the plant is going through the yarovization stage in solutions of those salts.

Description of the Experiment

The variety of winter wheat Lutescens 9329 was used in the experiment. In one case, seeds were yarovized for 52 days, in other words passed through the process of complete yarovization, necessary for that variety, while, in another case, the seeds were yarovized for 35 days. Part of the seeds of the yarovized wheat was soaked in a solution of KCl; another part, in NH_4NO_3; and still another, in NaH_2PO_4. The concentration of the solutions was as follows: KCl, 0.075 g per 18 cm³; NH_4NO_3, 0.12 g per 18 cm³; and NaH_2PO_4, 0.09 g per 18 cm³ of liquid. The seeds were moistured with water in a quantity of 35% of the weight of seeds. The weight of the seeds of each sample, 50 g. Different combinations of mineral elements from separate salts to ternary ones were applied. The seeds of winter wheat were yarovized, at a temperature of 0—3° for 52 and 35 days in solutions, mentioned above, according to the scheme of the experiment.

The yarovized control was kept in distilled water. After yarovization. the seeds were sown in vegetative vessels, no nutrition substances being added to the soil.

Seeds, yarovized in distilled water, were sown in vegetative vessels, usual quantities of nutritive substances being added to the soil, i. e. 0.5 g of active susbstance into each wessel (6 kg of soil). The mineral salts used were the same as had been employed in yarovizing the seeds in solutions, with the exception of nitrogen, which was added as $(NH_4)_2SO_4$. After thinning, 10 plants were left in each vessel.

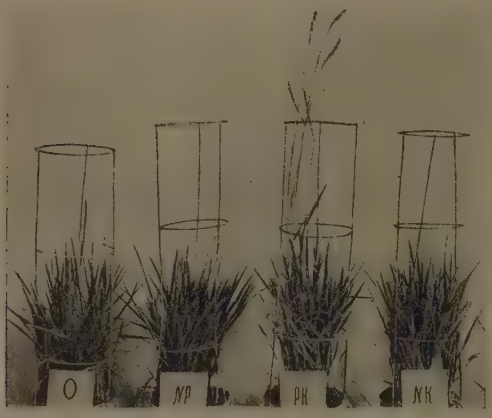

Fig. 1.

Phenological observations on the growth and development of wheat carried on throughout the vegetative period gave an expressive picture of the behavior of the plants experimented. No stalk-formation took place in the unyarovized wheat, regardless of the quality and quantity of mineral nutrition. The development of wheat yarovized for 52 days in solutions of mineral salts, differed in a marked degree from the development of wheat yarovized for 35 days in the same solutions.

A faster advancement of the phases of development and a more rapid ripening of seeds was observed in the first case. The development of the winter variety of wheat, whose seeds were yarovized for 35 days in a medium of mineral elements, differed from the control, which was yarovized during the same number of days in distilled water. A positive action of mineral elements took place in this case, an accelerating action of PK on the development of wheat in particular. Fig. 1 plainly confirms this statement. It shows PK to be a promotor of the yarovization process. Fig. 2 demonstrates the development of wheat, the seeds of which were yarovized for 52 days in solutions of mineral salts.

Fig. 2.

Of the double elements used in a 52 day yarovization NK exerted the most favourable effect on the development. It should be noted that the one factor of mineral nutrition cannot replace the entire complex of factors, necessary for passing through the stage of yarovization; so, in passing through the full term of yarovization, wheat eared in all the variants—fact, which did not take place when the yarovization lasted 35 days. This can be seen by comparing Fig. 1 with Fig. 2.

In the table given below the results are brought together of the effect of different mineral elements on wheat growth for the 52 day and 35 day yarovization respectively.

Table 1

Data on the Character of Development and on Grain Crop of Yarovized Wheat

Scheme of the experiment	Seeds of winter wheat Lutescens 0329 yarovized during 52 days in mineral salt solutions						%	Scheme of the experiment	Seeds of winter wheat Lutescens 0329 yarovized during 35 days in mineral salt solutions		
	Number of plants on August 2 at the stage of			Number of ears at the end of the experiment		Grain crop			Number of plants on August 2 at the stage of		
	Stalk-formation	Ea-ring	Flow-ering	Total	Ripe	Grams			Stalk-formation	Ea-ring	Flow-ering
	Average								Average		
O . . .	4	1	—	9	4	3.31	100	O . . .	—	—	—
N . . .	6	2	—	9	5	3.23	91.54	N . . .	—	—	—
P . . .	4	2	2	8	5	3.42	103.32	P . . .	—	—	—
K . . .	16	12	9	15	14	8.28	250.32	K . . .	—	—	—
NP . . .	9	3	2	11	8	6.65	200.90	NP . . .	—	—	—
NK . .	12	6	5	13	10	6.48	195.77	NK . .	—	—	—
PK . .	5	4	3	10	5	—	—	PK . .	4	2	1
NPK . .	9	8	4	14	9	6.55	197.90	NPK . .	—	—	—

The table shows an entirely different rate of development in yarovized wheat, according to the various rations of mineral elements and the length of the yarovization period. The recorded course the development stages are going on is maintained during further observations on the development of plants.

Of individual elements of mineral nutrition it is potassium which is especially distinguished by its effectual action on the development, yield, and acceleration of the ripening stage of the plant. The difference in the time of ripening, in comparison with the yarovized control, amounted to a month.

It should be remarked that the specific accelerating action of potassium on the ripening of cereals and other cultures, which had been observed in a series of experiments by other authors, as well as in our own, was also manifested in this case, when soaking the seeds of wheat in a solution of a potassium salt during the stage of yarovization.

One may say that seeds, having been soaked in a solution of a potassium salt receive a kind of an impulse, which manifests itself later in accelerating the period of ripening.

Of other elements, the accelerating influence of the group NK was the greatest.

The introduction of mineral fertilizers in our experiment had no considerable effect on the development of the yarovized wheat. One may conclude, on the basis of the foregoing, that the introduction of mineral elements at a definite stage of development promotes the process of yarovization, taking place at the point of growth of the stem. The character of this action has not yet been ascertained. Application of the elements of mineral nutrition has its advantage over the other methods of accelerating the process of yarovization (as ethylene, ether and others) worked out by some investigators [Eremenko (3)] in its influence being double; on one hand, it accelerates the

phases of development, on the other it has an advantageous effect on the yield of the crop.

The data obtained on the accelerating action of the elements of mineral nutrition, that of potassium in particular, upon the various phases of development and the process of yarovization, give additional means for the further study of this problem in vegetative, as well as in field conditions.

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PLANT ANATOMY

THE ENDOSPERM STRUCTURE OF THE PRINCIPAL SELECTION VARIETIES OF WHEAT OF THE USSR

By M. S. JAKOVLEV

(Communicated by N. I. Vavilov, Member of the Academy, 16. XI. 1937)

A detailed investigation of the endosperm structure of wheat revealed the presence therein of plastid and chondriosomal starch granules. Their shape and the mutual relation between them are responsible for the so-called endosperm mosaic pattern, characteristic of the principal groups of wheats (1).

It has been pointed out (2) that chondriosomal starch appears at a definite moment of the development of the grain. At the stage of milk ripeness, large, clearly distinguishable starch granules are deposited. This is the first phase of the grain's development. Then follows the second phase, during which a deposition of chondriosomal starch is observed to occur. Hence it is clear that plastid and chondriosomal starches play a very important part in the process of grain formation. These considerations will help to understand the endosperm structure.

Owing to differentiation between the two principal types of starch (plastid and chondriosomal), it has for the first time become possible to reveal characteristic features in the structure of the endosperm of a number of varieties. With this end in view, we investigated: 1) the soft winter wheats: Ukrainka (var. *erythrospermum* Körn.), Zaria 2537/64 (var. *erythrospermum* Körn.) and Hostianum 237 (var. *hostianum* Clem.); 2) the soft summer wheats: Sarrubra (var. *alborubrum* Körn.), Lutescens 062 (var. *lutescens* Körn.) and Balaganka (var. *ferrugineum* Al.); 3) the hard wheats: Hordeiforme 0432 and 027 (var. *hordeiforme* Host.) and Melanopus 069 (var. *melanopus* Al.).

The structure of the endosperm tissue has been studied on microsections made at the middle of the grain in three zones: I—the central zone; II—the one intermediate between the peripheral and the central parts; III—the peripheral zone, lying immediately below the aleurone layer. These zones correspond approximately to Cobb's first, third, and fifth zones (5), for which the gluten content has been reported to be 7.4, 9.5 and 16.5% respectively. However, Cobb does not describe the starch contained in these zones. Papers of other authors (4) contain some mentions of microscopic investigations of starch granules. Yet in all these investigations starch is treated as an inert reserve substance, its bearing on the general structure of the endosperm being disregarded. This results in starch being erroneously looked upon as the principal component of the endosperm. Even the shape of the granules, to say nothing of their mosaic pattern, escaped the investigator's attention. Only in solutions of sugars (2) it has been possible to observe the true picture

of the arrangement of starch granules in the endosperm. In our researches, a 20 per cent solution of saccharose was used. Good results were likewise obtained with concentrated (100%) solutions. The starch granules were not observed to leave the cells, and the protein was well visible. When the viscous gluten was present, its filaments were also clearly noticeable.

The endosperm of a wheat grain is a complex system of two principal components, starch and protein, which are in their turn far from homogeneous. All this makes an anatomical description of the endosperm very difficult. Only by applying very elaborate methods and by differentiating between

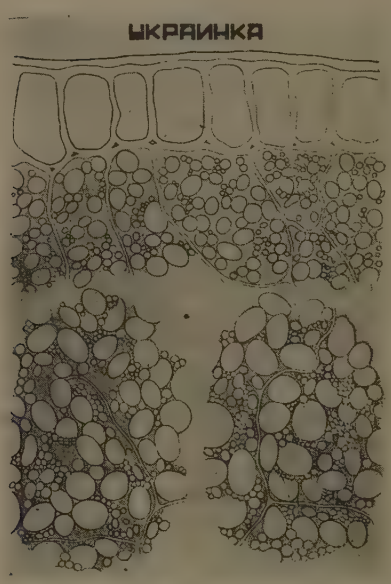

Fig. 1.—Ukrainka.

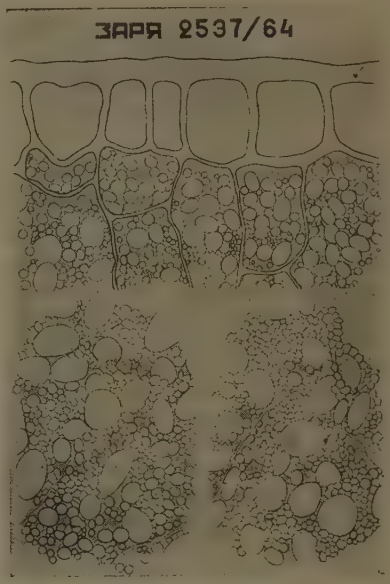

Fig. 2.—Zaria 2537/64.

the two principal types of starch, it has been possible to reveal some characteristic features in the structure of the endosperm cells of certain varieties.

As regards the varieties possessed of high baking qualities, such as Ukrainka, Sarrubra and Hordeiforme 0432 (Fig. 1), the endosperm structure in zones I, II and III may be characterized as follows:

I (the lower right-hand fragment of each figure).

The endosperm consists mainly of round grains of plastid starch, protein intercalations being clearly marked. Angular granules of chondriosomal starch occur in the form of small clusters.

II (the lower left-hand fragment of each figure).

The endosperm differs but little from that of the central zone: the only difference is that the protein intercalations are larger and the clusters of chondriosomal starch granules are correspondingly smaller.

III (the upper fragment of each figure).

The endosperm is very typical. Angular chondriosomal starch is completely absent. The size of the plastid starch granules is smaller. Strong deposition of protein is observed to occur.

Such is the general picture of the endosperm structure of the wheats in question. We are now going to give a description of another group of wheats, whose baking qualities contrast sharply with those of the first group. To this group belong Zaria, Balaganka and Hordeiforme 027 (Fig. 2).

I. The endosperm consists chiefly of chondriosomal starch granules. The protein intercalations are of a considerably smaller thickness as compared with those observed in the first group.

II. Chondriosomal starch predominates. The protein intercalations markedly increase.

III. The endosperm is characterized by the presence of chondriosomal starch granules, which either occur as individual clusters or surround the plastid starch grains.

The varieties Hostianum 237, Lutescens 062 and Melanopus 069 occupy an intermediate position as regards the distribution of the two types of starch.

It is considered that the best criteria of the baking qualities of a flour are the bulk of the bread and, partly, the porosity of the latter([3]). As regards the bulk of the bread, high values were shown by Ukrainka, Sarrubra and Hordeiforme 0432, viz. 449, 467 and 436 respectively. As a general rule, these varieties are distinguished by the presence of large protein intercalations, absence of chondriosomal starch in peripheral layers and its presence in deeper layers of the endosperm in the form of small clusters (see Figs 1, 2). Low values were shown by Zaria, Balaganka and Hordeiforme 027, viz. 391, 408 and 237 respectively. The peculiarities of the endosperm structure of these varieties are very characteristic. Chondriosomal starch is present here in great quantities, occurring even in peripheral layers of the endosperm. The plastid starch granules are often surrounded with minute angular starch granules; the protein intercalations are not so considerable as in the first group of wheats.

It has been established that the starch of different wheats does not glutinize in the same way. Its properties in the dough, where it is a viscous, strongly hydrated gel, differ in the different wheat varieties. The higher the viscosity of the starch, the lower the quality of the bread ([6]). This important phenomenon seems to be associated with the existence of two types of starch granules. The physical properties of the chondriosomal starch granules differ from those of the plastid starch granules. It is to be expected that the starch of the varieties whose endosperm contains greater quantities of chondriosomal starch, will be found to be more viscous.

The gluten content in the grain endosperm is the factor determining the colloidal properties of the dough. The gliadin and glutenin contained in the gluten differ as to their qualities. Their colloidal properties vary from variety to variety and with the conditions of growth. We have observed that, in microsections of the grains of Zaria, Balaganka and Hordeiforme 027, i. e. those varieties which contain great quantities of chondriosomal starch, filaments of viscous gluten always occur. Thus, a certain connection seems to exist between chondriosomal starch and the characteristic properties of gluten. The differences in the colloidal properties of chondriosomal starch (greater viscosity, increased susceptibility to diastatic activity) and the presence of highly characteristic gluten (viscous filaments) set up conditions resulting in a bread of poor quality. Consequently, the significance of chondriosomal starch is very great, and this should be taken into consideration in evolving new, high-quality varieties. The endosperm structure of related varieties, as, for instance, Ukrainka and Zaria (var. *erythrospermum* Körn.) or Hordeiforme 0432 and 027 (var. *hordeiforme* Host.), may display clearly marked anatomical peculiarities even in grains quite similar as to their size, shape and flintiness, the principal criterion

being in this case the form of chondriosomal starch and degree of its accumulation.

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MYCOLOGIE TECHNIQUE

SUR L'ÉCHAUFFURE DU BOIS DE HÊTRE

Par A. A. YATSENKO-KHMÉLEVKY

(Présenté par A. A. Richter, de l'Académie, le 13. XII. 1937)

L'altération la plus fréquente du bois de hêtre après l'abatage, c'est la maladie généralement connue sous le nom de «l'échauffure» («Erstickung»— des auteurs allemands). Selon la plupart des auteurs qui ont étudié cette question ([1], [2], [3], [4], [5], [6]), l'échauffure est due à l'installation dans le bois coupé encore vivant, des champignons destructeurs du bois. Les premiers stades de l'altération sont caractérisés par le brunissement du bois. Plus tard la décoloration commence, le bois se couvre de taches blanchâtres ou jaunâtres, entourées par des lignes noires («bois en marbre»). Dans cet état le bois perd la plupart de ses qualités techniques et ne peut être utilisé que comme bois de chauffage.

Parmi les champignons provoquant l'échauffure peuvent être notés, pour les forêts de l'Europe occidentale, *Stereum purpureum*, *Hypoxylon coccineum*, *Schizophyllum commune*, *Bispora moniloides* etc., et pour les forêts caucasiennes, *Stereum hirsutum*, *Fomes fomentarius*, *Bulgaria polymorpha* et d'autres.

Dans les conditions climatiques de la Transcaucasie, caractérisées par la température et l'humidité de l'air élevées, l'échauffure peut être considérée comme l'altération la plus dangereuse du bois de hêtre (*Fagus orientalis* Lip.), qui est l'essence forestière la plus répandue des forêts de la Géorgie, d'Azerbaidjan et d'Arménie. Les pertes causées par la dépréciation du bois due à l'échauffure sont très considérables et c'est pourquoi l'Institut des Recherches Forestières de la Transcaucasie a entrepris une étude spéciale de ce phénomène. Ces études ont été poursuivies par l'Institut durant les années 1932—1935*.

La méthode employée dans nos investigations a été la suivante: aux termes différents, on abattait les arbres et de chaque arbre on prenait une grume longue de 4 mètres. Au moment de l'abatage on prenait des échantillons qui servaient de contrôle. Les grumes expérimentales étaient laissées ou simplement gisant par terre, comme c'est d'usage dans la pratique forestière de la Transcaucasie, ou étaient placées sur des perches. Une part des grumes a été privée de l'écorce. Dix, vingt, trente, quatre-vingt-dix et cent quatre-vingt jours après l'abatage deux des grumes expérimentales étaient sciées en pièces. Les échantillons (en forme de petits tronçons) étaient pris

* Dans ces travaux ont pris part, outre l'auteur, D. K. Antonov et H. L. Konuchevska.

des deux bouts de la grume et à la distance de 0.5, 1.0, 2.0, 3.0 et 3.5 m du bout inférieur de la grume. En tout, on prenait chaque fois 7 tronçons, ce qui permettait de recevoir le tableau complet de tous les changements survenus dans la grume.

Chaque tronçon était dessiné schématiquement, pour noter les changements de couleur du bois, après quoi, avec une hâche, on en prenait des échantillons pour l'étude microscopique du bois. Les échantillons étaient pris au bord du tronçon, avec l'écorce adjacente, au milieu du rayon et, de plus, là où le tronçon présentait quelques modifications de la couleur (en comparaison avec le bois normal).

Durant les travaux des années 1934 et 1935, simultanément avec la prise des échantillons pour l'étude microscopique, on prélevait sur chaque tronçon des échantillons pour déterminer l'humidité du bois.

Le total des échantillons examinés au microscope est donné dans la table suivante.

Afin que le résultat de nos recherches soit fondé sur un plus grand nombre de données, nous avons examiné en outre quelques centaines de grumes de hêtre, sciées à la scierie de Mzkhet (Géorgie).

Année, lieu et l'altitude au-dessus du niveau de la mer / Saison	1932 Bacouriani, 1400 m		1933 Tkvartchéli, 300 m		1934 Vardisoubani, 1100 m		1935 Akhmeti, 800 m	
	Grumes	Échant.	Grumes	Échant.	Grumes	Échant.	Grumes	Échant.
Printemps	14	196	8	112	—	—	—	—
Été	22	308	7	98	20	360	10	140
Automne	18	252	—	—	—	—	2	28
Hiver	10	140	3	42	—	—	2	28
Total	64	896	18	252	20	360	14	196

Furent étudiés tout le long du travail: grumes, 116 pièces; échantillons, 1704 pièces.

Le nombre considérable des échantillons et leur étude détaillée nous a permis de fixer quelques particularités dans le processus de l'échauffure, qui, comme il semble, ont été inaperçues par les auteurs précédents.

En effet, tous les auteurs qui ont consacré des études spéciales à l'échauffure du bois de hêtre, ont examiné ce phénomène au point de vue de l'activité des champignons provoquant l'altération du bois, tandis que la réaction physiologique du bois à l'attaque des champignons n'était examinée qu'indirectement. Néanmoins, on peut affirmer que cette réaction du bois est un facteur très important, qui détermine le caractère du processus. Gäumann, l'éminent phytopatologue suisse, qui, dans son article (6) paru il n'y a pas longtemps, a donné l'analyse la plus complète du phénomène de l'échauffure, s'attache principalement à l'étude de l'intensité de l'échauffure dans les saisons différentes, ce qui est lié avec des changements périodiques dans la substance ligneuse. Le mécanisme de la résistance du bois à l'introduction du champignon est décrit par cet auteur avec moins d'ampleur.

Au contraire, dans notre travail l'attention fut portée principalement à l'étude de la propagation de l'échauffure dans la grume et de la réaction du bois à l'attaque des champignons. Les problèmes d'ordre mycologique (la mycoflore de l'échauffure etc.) ont été mis de côté, car nous sommes assez bien renseignés sur ce point par les travaux des auteurs cités plus haut.

La propagation de la coloration brune du bois (qui peut être considérée comme l'indice extérieur principal de l'échauffure) peut être présentée comme il suit: l'échauffure commence des deux bouts de la grume. Les premiers indices de l'altération, c'est l'apparition de nombreuses taches brunâtres bien visibles sur le bois fraîchement coupé. Peu à peu ces taches se fusionnent et le bout de la grume devient entièrement brunâtre. Cette coloration se propage dans la masse de la grume en forme d'un cône (fig. 1) dont la base est le bout de la grume. Si la grume possède un «faux-coeur» (faux bois parfait), la propagation de la coloration brune conserve la même forme, mais

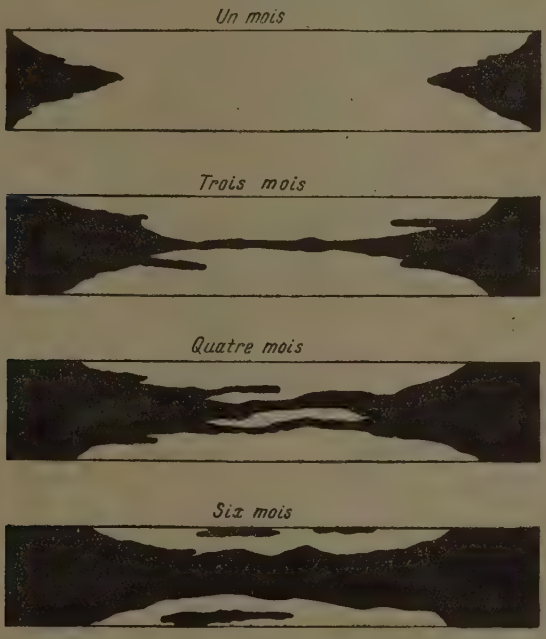

Fig. 1. Schémas indiquant la propagation de la coloration brunâtre du bois dans la masse de la grume de hêtre, longue de 4 m, et laissée dans le taillis un mois, trois mois, quatre mois et six mois.

entoure le faux coeur. A peu près trois mois passés après la coupe de l'arbre, les deux cônes partant des deux bouts de la grume, s'unissent par leurs sommets fortement allongés et dans six mois tout le bois de la grume acquiert une coloration brunâtre. Vers ce terme dans les deux bouts de la grume apparaissent les signes macroscopiques de la carie du bois—les taches jaunes et blanches, les lignes noires etc. Si l'écorce est endommagée par la chute de l'arbre ou la grume reste couchée sur la terre moite (ce qui mène aussi à l'altération de l'écorce), la contamination commence par ces écorchures aussi et se propage généralement le long de l'axe de la grume.

La propagation de l'échauffure de la manière décrite est une règle générale, qui a été observée dans toutes les grumes expérimentales (ainsi que dans toutes les grumes que nous avons étudiées dans la scierie de Mzhet). La marche de l'échauffure peut être accélérée, ou, par contre, ralentie en dépendance de la saison de la coupe, de l'altitude et des conditions météorologiques de l'année, mais le caractère de sa propagation reste toujours le même.

L'analyse microscopique des échantillons pris tout le long de la grume (selon le schéma donné plus haut) a montré que les portions brunies du bois sont caractérisées par une formation énergique de thylles, par la transformation des matières plastiques dans des substances d'une composition chimique indéterminée, de couleur et de configuration différentes, et par la présence (assez rare) des gyphes des champignons.

Avec cela, il faut noter que le nombre de thylles diminue de la périphérie (aubier) vers le centre. Le bois qui a conservé sa coloration normale, au contraire, ne diffère presque pas du bois fraîchement coupé—les thylles y sont absentes ou rares et les matières plastiques ne manifestent presque pas de changements*.

La détermination de l'humidité du bois dans les différentes parties de la grume donne un tableau assez caractéristique. L'humidité du bois des

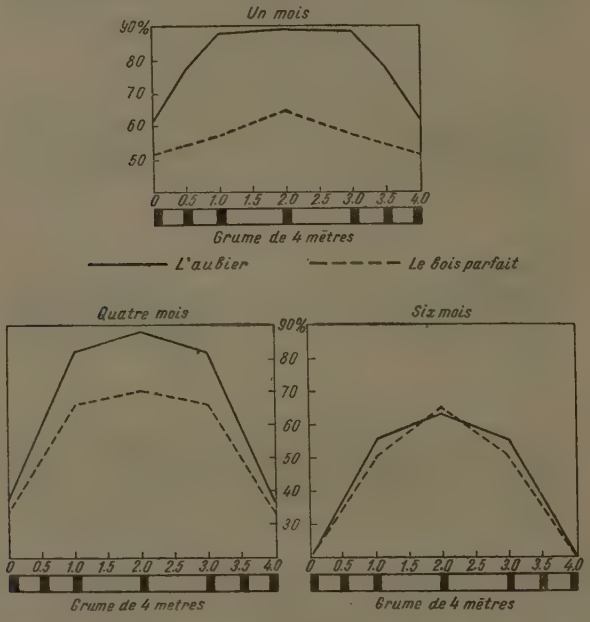

Fig. 2. Courbes indiquant la distribution de l'humidité du bois (en pour cent du poids absolument sec) dans la masse de la grume de hêtre, longue de 4 m et laissée dans le taillis un mois, quatre mois et six mois.

arbres sur pied est à peu près égale à 80 pour cent (du poids absolument sec). Comme on peut juger en examinant les courbes (fig. 2), entre la propagation de la coloration brune du bois et son humidité existe une relation: les bouts de la grume surtout dans les parties centrales, possèdent une humidité bien moindre que le milieu de la grume, qui conserve bien longtemps

* La réaction traumatique du bois ([9], [10]), sans aucun doute, a lieu dans ce cas-là aussi. Néanmoins, dans les parties brunies du bois, elle est complètement masquée par le processus bien plus énergique—l'échauffure, tandis que dans les parties de la grume qui ont conservé la couleur du bois normal, la réaction traumatique a une marche assez lente, et c'est seulement dans les échantillons pris après quatre-vingt-dix et cent quatre-vingt jours qu'on peut noter une différence bien exprimée entre le bois fraîchement coupé et le bois des grumes. Ce ralentissement de la réaction traumatique est dû, probablement, à la conservation de l'humidité élevée du bois, phénomène noté dans notre travail sur la réaction traumatique du bois de saule et de pistachier([10]).

une humidité élevée qui ne diffère presque pas de l'humidité du bois de l'arbre croissant.

Tout ce qui a été exposé plus haut permet de se présenter le processus de l'échauffure de la manière suivante: après la coupe de l'arbre et son tronçonnage, sur la surface nue des bouts de la grume se déposent les spores des champignons, toujours abondantes dans les forêts. En germant, les champignons pénètrent dans la masse du bois et, sous l'influence des gyphes, les cellules vivantes du bois produisent des thylles nombreuses et dans ces mêmes cellules s'accomplit la transformation des matières plastiques et, probablement, du protoplasme en produits qui donnent au bois échauffé sa couleur brunâtre. Ces processus biochimiques sont très complexes ([7], [8]). En tout cas, les produits qui se forment dans la cellule vivante sous l'influence des gyphes des champignons diffèrent de «la matière traumatique du bois» ([9], [10]), comme le démontre leur forme et leur coloration.

La formation énergique des thylles, bouchant hermétiquement les vaisseaux, ralentit le séchage du bois, ce qui est favorisé par la conservation de l'écorce. Ce ralentissement est plus considérable dans l'aubier, où les cellules vivantes sont plus actives et où, par conséquent, les thylles sont plus nombreuses. L'humidité élevée du bois, comme l'ont montré Münch et Bavendamm ([11]), entrave la propagation des champignons et c'est pourquoi la croissance des gyphes est moins intense dans les couches périphériques de l'aubier et augmente vers le centre où l'humidité du bois est moindre parce que, primo, ces couches-là dans l'arbre vivant sont caractérisées par une moindre humidité et, secundo, parce que la formation moins intense des thylles favorise un séchage plus rapide. Cette répartition de l'humidité dans les différentes parties de la grume explique la propagation de l'échauffure en forme de deux cônes.

Les processus subséquents de la destruction du bois (l'apparition des signes macroscopiques de la carie du bois) ont lieu dans le bois mort et, par conséquent, n'ont pas de rapport avec l'échauffure au sens strict du mot.

En tronçonnant la grume à quelque distance du bout, on observe le bois échauffé (bruni) en forme d'un cercle ou d'un anneau entourant le faux-coeur. Ce fait a amené Fljerov et Chemakhanova ([4]) à la conclusion que les champignons contenus dans le faux-coeur jouent un rôle dans le phénomène de l'échauffure. Néanmoins, il semble que cette assertation n'est pas exacte, car la même propagation peut être observée dans les grumes privées du faux-coeur.

Dans les cas où l'écorce de la grume est enlevée, le mécanisme du séchage du bois, décrit plus haut, disparaît, grâce à l'évaporation de la surface mise à nu, et la propagation de l'échauffure perd sa régularité.

Les méthodes communes de la défense du bois de hêtre contre l'échauffure, telles que l'enlèvement de l'écorce ([1]), le badigeonnage des découpes de la grume avec des substances antiseptiques ([1], [3]) ou le séchage physiologique ([12]) n'ont pas donné de résultats positifs dans nos expériences. C'est pourquoi, la meilleure méthode d'obtenir un bois sain est, comme nous l'avons indiqué il y a trois ans, «une telle organisation de l'exploitation forestière qui aurait permis de débiter le bois de hêtre dans une scierie dans 4—5 jours après l'abatage» ([13]). Néanmoins de telles conditions ne peuvent se présenter toujours. Dans les cas où l'on est obligé de conserver les grumes dans la forêt un temps plus ou moins considérable, le plus rationnel, c'est, selon notre opinion, de conserver dans le bois une humidité élevée. Dans ce but, nous avons essayé de badigeonner les découpes des grumes avec du bitume dissout dans de la benzine. Avec ce procédé le séchage du bois devient très lent et les résultats obtenus ont été excellents. Il faut tenir compte encore que le bitume lui-même n'est pas un antiseptique et que la

conservation du bois dans ce cas doit être expliquée exclusivement par l'humidité élevée du bois. Un procédé analogue est employé en France, où on pratique le badigeonnage des découpes avec du carbonyle (ou du créosote) (3), toutefois sans l'idée de la nécessité de conserver le bois en état d'humidité élevée.

Pour finir, il faut noter que les formes de la propagation des champignons similaire à la propagation décrite plus haut, doit être observées probablement dans les grumes des autres feuillus, aptes à la formation des thylles. Les études subséquentes permettront de vérifier cette assertation.

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DEVELOPMENT MECHANICS

STIMULATION OF METAMORPHOSIS IN ASCIDIAN LARVAE

By L. N. ŽINKIN

(Communicated by N. V. Nasonov, Member of the Academy, 7. XII. 1937)

In experiments on larvae of Cyona evidence was presented by P. Weiss (1) that in sea water to which thyroxin had been added the percentage of metamorphosed larvae is larger than in normal sea water. By these experiments P. Weiss sought to prove that the factors stimulating metamorhosis in Ascidian larvae are the same as in Amphibia and it is known that the prime factor influencing metamorphosis in the latter group is the hormone of the thyroid gland. Similar experiments were carried out by W. Brodway 1936 (2) who studied on larvae of *Clavellina* the action not only of thyroxin but also of inorganic iodine, neutral red, various salts as well as the effect of hypotony. In all these experiments Brodway found stimulation of metamorphosis, that is 2 and 24 hours after the beginning of the experiment more larvae had metamorphosed than in the control. Thence it is to be concluded that thyroxin, has no specific action on the metamorphosis of Ascidian larvae.

Our present research were undertaken with the view of studying the mechanism of metamorphosis in *Ascidia*. In these experiments the effect produced by vital stains and various poisons on the metamorphosis of the larvae of the colonial Ascidian *Botryllus schlosseri* (Pallas.) was examined. The investigations were accomplished in the summer of the year 1937 in the Sebastopol Biological Station of the Academy of Sciences.

Colonies of *Botryllus* removed from piles were brought to the Station and placed in aquaria; next, small portions of the colonies were placed in crystallizing dishes and from these larvae which had left the colony were picked. The eggs of *Botryllus* develop in the peribranchial cavity where they remain up to the formation of the free-swimming larval form, at which stage they leave the colony. In all the individuals of the colony the development of the sexual products is more or less synchronous, so that all the larvae which leave the colony are in the same stage of development.

The free-swimming larvae were picked and placed in dishes containing sea-water to which poison, resp. stain, had been added. The technique of the experiment was as follows: the basal solution of the stain, resp. of the poison, was diluted with distilled water and several tenth parts of 1 cm^3 added to 10 cm^3 of sea-water. The liquid was thoroughly mixed and then the larvae were placed in it. Simultaneously control experiments, with normal sea-water, were carried out.

The description of the experiments must be preceded by a short exposition of the normal metamorphosis.

After a certain period of time the free-swimming larvae of *Botryllus* attach themselves to some object under water and then their metamorphosis begins. Frequently however the metamorphosis may set in without the attachment of the larvae taking place. The metamorphosis begins with papillae retracting at the anterior end of the larvae, and with a simultaneous swelling of the 8 ampullae similarly located (Fig. 1, *a*, *b*, *c*). As soon as these processes begin at the anterior end, reduction of the tail at the posterior end sets in,

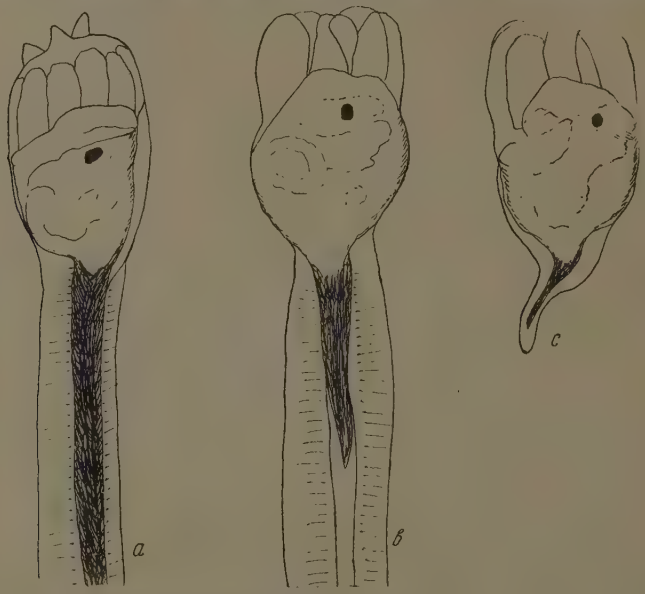

Fig. 1. Consecutive stages of the metamorphosis of *Botryllus* larvae. *a*—normal larvae; *b*—larvae just beginning to metamorphose; *c*—larvae in which retraction of the tail get finished.

the latter process lasting about an hour. The movements of the tail cease before the retraction has set in. The larvae being but feebly transparent, the metamorphosis of definitive organs is difficult to observe. Later on, in the course of metamorphosis the caudal region shifts, the siphons are formed, elongated provisory gill clefts become visible. But these changes make their appearance several hours after the beginning of metamorphosis.

The duration of the larval free-swimming stage previous to metamorphosis varies greatly. Counts were made of larvae undergoing metamorphosis at a period of time of from 1 to 3 hours after the larvae had left the colony. The results are given in Table 1.

Table 1

Time	0—1 h.	1—2 h.	2— 3 h.	Total number of larvae examined
Number of larvae in which metamorphosis had set in	21	40	54	129
Per cent of all larvae examined . . .	16.2	31.0	42	—

Very few larvae start metamorphosis after a lapse of 24 hours or later. In most larvae metamorphosis sets in at a period of from 2 to 3 hours.

These findings coincide in full with the data of Grave and Woodbridge (3); who have made a detailed study of the biology and morphology of the *Bo tryllus* larvae. Most of the larvae, in which metamorphosis has not set in soon (in the first three hours) after their leaving the colony, remain unchanged and do not live long.

The substances examined in their effect on the *Botryllus* larvae were: neutral red, brilliant cresyl blue, methylene blue, cyanide of potassium, potassium bichromate, potassium permanganate, strychnine and cocaine, further the action of hypotony was likewise studied.

Beside the three vital stains, strychnine and potassium bichromate were found to produce a stimulating action, while as regards the other substances the effect proved to be negative or, may be, the required concentration had not been found.

In the present publication only the action of the vital stains with which the most distinct results were obtained will be discussed.

As shown in Table 2 neutral red has been employed in three different concentrations. The strongest stimulating action on metamorphosis was found to be produced by 0.002%. In larvae placed in the dyestuff metamorphosis sets in almost immediately: of the 22 larvae involved in the experiments 15, that is 68% of all the larvae, began to metamorphose after 10 minutes. In three larvae metamorphosis commenced after 30 minutes. The other larvae evidently do not live. This concentration of neutral red is, it would seem, poisonous for the *Botryllus* larvae as the larvae which had begun their metamorphosis lived but a few hours. The 0.004% concentration is less favourable, while the next used, 0.008%, is positively noxious. In it only four of all the larvae started the retraction of their tail, but the process did not reach to its conclusion. The other larvae died without having begun metamorphosis.

Table 2

Stain	Concentration %	Number of larve beginning their metamorphosis after						Total number of larvae in metamorphosis	Number of larvae unchanged
		10 min.	20 m.	30 m.	1 hour	2 h.	3 h.		
Neutral red	0.002	15	15	18	18	18	18	22	4
	0.004	7	14	14	14	14	14	18	4
	0.008	3	4	4	4	4	4	21	17
Brilliant cresyl blue	0.002	8	17	17	17	17	17	23	6
	0.004	10	11	11	11	11	11	16	5
	0.008	17	19	20	20	20	20	21	1
	0.016	15	23	23	23	23	23	27	4
Methylene blue	0.0005	6	8	8	8	8	8	12	4
	0.001	6	10	11	11	11	11	15	4
	0.002	10	18	20	20	20	20	21	1
	0.004	3	6	6	10	10	10	11	1
	0.008	0	0	0	0	0	0	18	18

Brilliant cresyl blue and methylene blue have about the same effect.

Brilliant cresyl blue, as shown in Table 2, produces a distinctly stimulating effect. In the four doses employed over 50% of larvae begin their

metamorphosis 10 minutes—that is nearly at once—after they have been placed into the solution. The action produced by methylene blue was somewhat more differentiated: used in 0.008% concentration the stain is too poisonous and the larvae die without coming to metamorphose. In a 0.004% concentration the larvae begin to metamorphose, but the process has a very slow course and is not concluded. It is in 0.002% that methylene blue has the most favourable effect. Speaking of the action of all the stains it is to be stated that, employed in certain doses, they distinctly stimulate metamorphosis. The data obtained are shown in Fig. 2 where time is plotted against the number of larvae (in per cent of the total number of larvae) in which metamorphosis had set in. The curve illustrates the action of neutral red in a 0.002% concentration, of brilliant cresyl blue in a 0.008% concentration and of methylene blue in a 0.002% concentration.

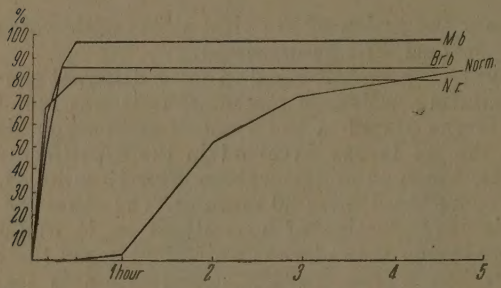

Fig. 2. Beginning of metamorphosis in normal *Botryllus* larvae. *Br. b.*—Metamorphosis of larvae in brilliant cresyl blue *N. r.*—Ditto in neutral red. *M. b.*—Ditto in methylene blue. *Norm*—in pure water.

Thus, placed in dyestuff solutions, most larvae begin to metamorphose in the course of the first half hour, while in the control the largest number of larvae begin to metamorphose after a period of from 1 to 3 hours.

The larvae just placed in the dyestuff solution show, under the microscope, that at their anterior end, between the papillae, a swelling arises which very soon comes to burst, whereupon retraction of the tail immediately sets in, i. e., the metamorphosis begins. The swelling is not always distinctly visible; however it is visible in most larvae placed in the stains employed. A certain deformation occurring at the anterior end precedes metamorphosis.

These observations can serve to explain the facts above mentioned.

Of the provisory organs the larvae possess a very well developed nervous system (4). At the anterior end of the larva lie three ganglia innervating the papillae and two ganglia innervating the attaching membrane. The stain, resp. the poison, which influences the anterior end, undoubtedly exert its influence upon the ganglia also. Immediately after the changes mentioned the tail begins to retract which means that the irritation is passed on from the anterior to the posterior end. The only system of organs connecting the posterior with the anterior end is the nervous system and it is this system which presumably causes the setting in of metamorphosis. This surmise is the more plausible as in insects it is the nervous system that at the present is considered to be responsible of the metamorphosis. Further experiments to be made in this direction are planned and we expect to obtain from them corroboration of our point of view here exposed.

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